Math Metods
Day 13

Today
I. Last Time
II. Normal Modes & the Wave Equation
III. General Solutions & Fourier Series

Last Time

- We studied the limit $N \to \infty$ of our model and found $\frac{p}{N+1} = \frac{x}{L}$, as well as
$$y_x(t) = y(x,t) = A\sin(n\pi\frac{x}{L})\cos(\omega_n t + \phi_n),$$
with
$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}.$$

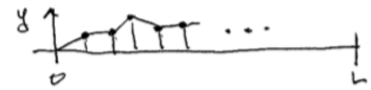

Figure 1: Our discrete model of transverse oscillations.
μ = linear mass density

- We derived the wave equation
$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$
for waves on a taut string and showed $v = \sqrt{T/\mu}$.

Figure 2: A taut string of length L.

Normal Modes and the Wave Equation

Normal Modes: The definition of a normal mode is that each position along the string oscillates at the same frequency. We can encode this definition in a good guess for the solution of the wave equation:
$$y(x,t) = f(x)\cdot\cos(\omega t + \phi).$$
Here $f(x)$ is an "amplitude function" that we don't know how to anticipate and which we will try to adjust to solve the wave equation.

Using this guess,
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x)\cos(\omega t + \phi)$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{dx^2}\cos(\omega t + \phi).$$
Putting these expressions into the wave equation gives:
$$-\omega^2 f(x)\cos(\omega t + \phi) = v^2 \frac{d^2 f}{dx^2}\cos(\omega t + \phi)$$
$$\Longrightarrow \frac{d^2 f}{dx^2} = -\left(\frac{\omega}{v}\right)^2 f$$

Simple Harmonic Oscillator Eq. (SHO)

So,
$$f(x) = A\sin\left(\frac{\omega}{v}x\right) + B\cos\left(\frac{\omega}{v}x\right),$$
and
$$y(x,t) = \left[A\sin\left(\frac{\omega}{v}x\right) + B\cos\left(\frac{\omega}{v}x\right)\right]\cos(\omega t + \phi).$$

But: we have boundary conditions! Recall that we said that the string was tacked down at both ends, then

$$(1) \quad y(0,t) = 0; \qquad (2) \quad y(L,t) = 0.$$

From (1), and since this boundary condition is supposed to hold for all times t,

$$B\cos(\omega t + \phi) = 0 \implies B = 0.$$

So,

$$y(x,t) = A\sin\left(\frac{\omega}{v}x\right)\cos(\omega t + \phi).$$

Then (2) implies

$$y(L,t) = A\sin\left(\frac{\omega}{v}L\right)\cos(\omega t + \phi) = 0.$$

We don't want $A = 0$, so require

$$\sin\left(\frac{\omega}{v}L\right) = 0 \implies \frac{\omega}{v}L = n\pi \qquad (n = 1, 2, 3, \dots),$$

or

$$\omega_n = \frac{n\pi}{L}v = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}.$$

n^{th} normal mode frequency

This is in perfect agreement with our limit of the coupled oscillators. The normal modes are

$$y_n(x,t) = A\sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t + \phi)$$
$$\omega_n = \frac{n\pi v}{L}; \qquad n = 1, 2, 3, \dots,$$

these are standing waves!

Note:

$$\omega_n = \frac{n\pi v}{L} \implies v = \frac{L\omega_n}{n\pi} = \frac{2L}{n}\frac{\omega_n}{2\pi} = \lambda f.$$

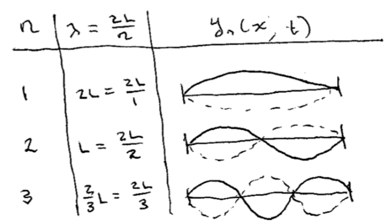

Figure 3: The first three normal modes.

BUT: Normal modes are not the most general motion of a string. Two nice examples that clearly don't give standing waves are:

- A plucked guitar string
- A hammered piano string

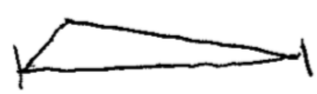
Figure 4: A plucked string has a rich initial shape, but no initial velocity.

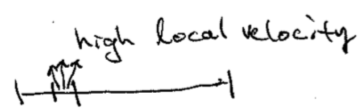

Figure 5: A hammered string has a rich initial velocity, but no initial deformation.

General Solutions & Fourier Series

The General Solution: The two pictures at right correspond to interesting initial conditions:

$$F(x) = y(x,0) \qquad \text{(given initial shape)}$$

and

$$G(x) = \dot{y}(x,0) = \frac{\partial y}{\partial t}(x,0) \qquad \text{(given initial velocity).}$$

Claim: The general solution is (as for coupled oscillators) a linear combination of the normal modes. Namely,

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi v}{L}t + \phi_n\right).$$

Our two special initial conditions require either

$$F(x) = y(x,0) \implies F(x) = \sum_{n=1}^{\infty} (A_n \cos\phi_n) \sin\left(\frac{n\pi}{L}x\right),$$

or

$$G(x) = \dot{y}(x,0) \implies G(x) = \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi v}{L} \sin\phi_n\right) \sin\left(\frac{n\pi}{L}x\right).$$

We have to pick A_n and ϕ_n so as to meet these two conditions. Can we do this? How?

We go to math. To do so, let's recast this as a question directly to mathematics: given $f(x)$, find constants b_n such that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \qquad \text{on} \quad (0 \le x \le L).$$

This expression is called the Fourier series expansion for $f(x)$.