Math Methods

Day 14

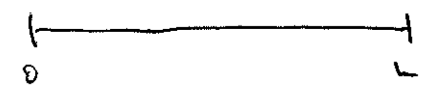
Today

I Last time

II Fourier Analysis

III Extended Domain

I • We studied Normal Modes on a taut string of tension T and length L:



$v = \sqrt{\frac{T}{\mu}}$

• Guessed the form of the normal modes

$$y(x,t) = f(x)\cos(\omega t + \phi)$$

and found

$$y_n(x,t) = A\sin\left(\frac{n\pi}{L}x\right)\cos\left(\omega_n t + \phi\right)$$

$$\omega_n = \frac{n\pi v}{L}\,; \quad n = 1, 2, 3, \ldots$$

• Began study of the general solution

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi v}{L}t + \phi_n\right)$$

subject to the initial conditions

$$F(x) = y(x,0) \Rightarrow F(x) = \sum_{n=1}^{\infty}\left(A_n\cos\phi_n\right)\sin\left(\frac{n\pi}{L}x\right)$$

and

$$G(x) = \dot{y}(x,0) \Rightarrow G(x) = \sum_{n=1}^{\infty}\left(-A_n\frac{n\pi v}{L}\sin\phi_n\right)\sin\left(\frac{n\pi}{L}x\right)$$

II <u>Fourier Series</u> expansion of $f(x)$:

Given $f(x)$, find constants b_n such that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \quad \text{on } (0 \le x \le L)$$

<u>Dirichlet's Theorem</u>: A function $f(x)$ admits a Fourier Series expansion provided it satisfies the "Dirichlet conditions":

(1) It has a finite # of maxima &

minima on the interval ($0 \le x \le L$ for us).

(2) It has a finite # of discontinuities on the interval.

(3) $\int_0^L |f(x)|\,dx$ is finite.

Remark: In this case $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ converges to $f(x)$ at every point where $f(x)$ is continuous; at discontinuities the series converges to the midpoint.

Thus,

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}b_m.$$

or

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx.$$

If $f(x)$ is known we can find the b_m by doing this integral.

Example 1: If $f(x) = 1$ then

$$b_n = \frac{2}{L}\int_0^L 1\cdot\sin\left(\frac{n\pi}{L}x\right)dx$$

How do you get the coefficients a_n?

Fourier's Trick: Multiply both sides by $\sin\left(\frac{m\pi}{L}x\right)$ $(m = 1, 2, \ldots)$ and integrate

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx$$

$$= \sum_{n=1}^{\infty} b_n \frac{L}{2}\delta_{nm}$$

where δ_{mn} is the Kronecker δ

$$\delta_{nm} = \delta_{mn} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{else.} \end{cases}$$

$$= -\frac{2}{L}\left(\frac{L}{n\pi}\cos\left[\frac{n\pi}{L}x\right]\Big|_0^L\right)$$

$$= -\frac{2}{n\pi}\left(\cos(n\pi) - \cos(0)\right)$$

$$\Rightarrow b_n = \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

n	b_n
1	$4/\pi$
2	0
3	$4/3\pi$
4	0
5	$4/5\pi$

so,

$$1 = \frac{4}{\pi}\sin\left(\frac{\pi x}{L}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi x}{L}\right) + \cdots$$

$$= \frac{4}{\pi}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + \cdots\right)$$

<u>Extension</u>: Does this series converge to 1 outside of $(0, L]$? No. Note: $\sum b_n \sin\left(\frac{n\pi x}{L}\right)$ is <u>odd</u>, and periodic, with period $2L$.

<u>Ex. 2</u>: Take $f(x) = x$ then

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{2}{L}\int_0^L x \frac{d}{dx}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right] dx$$

$$= \frac{2}{L}\left\{-\frac{Lx}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi}{L}x\right) dx\right\}$$

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$$= \frac{2}{L}\left\{-\frac{L^2}{n\pi}(-1)^n + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{L}x\right)\Big|_0^L\right\}$$

$$= \frac{2L}{n\pi}(-1)^{n+1}$$

n	b_n
1	$2L/\pi$
2	$-L/\pi$
3	$2L/3\pi$

Then

$$x = \frac{2L}{\pi}\left(\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \cdots\right)$$

Choose for simplicity $L = \pi$

$$x = 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \cdots\right)$$

A wonderful tool for numerous applications.

Could I write

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)\ ?$$

Yes. This is called a cosine series (the last a sine series). We have

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) \quad \text{again } 0 \le x \le L$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right) dx \quad \text{for } n \neq 0$$

and

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{or} \quad \frac{a_0}{2} = \frac{2}{L}\int_0^L f(x)\,dx.$$

<u>But</u>: Sine series extends (outside $0 \le x \le L$) to an <u>ODD</u> function (w/ period $2L$) where as cosine series extend to an <u>EVEN</u> function (still period $2L$).

<u>III</u> Let's define Fourier Series on $-L \le x \le L$. If $f(x)$ is odd, can use a sine series, if $f(x)$ is even use cosine series. We'll do this next time.