Math Methods

Day 15

Today

I Exam Discussion

II Last time

III Zak's Guest Lecture: Putting together the Fourier Expansion

I We went over peoples' questions from the exam.

II • We reviewed Dirichlet's theorem: $f(x)$ has a Fourier series if

(1) it has a finite # of max & mins.

(2) " " " " " of discontinuities

(3) $\int_0^L |f(x)|dx$ is finite

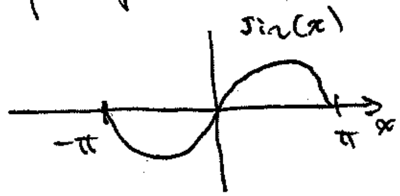
• We used Fourier's trick to derive

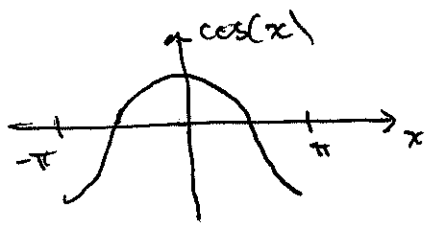
$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx.$$

III Last time we studied the sine function Fourier Series with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Notice that both the sine and cosine functions have special properties





Notice that

$\sin(-x) = -\sin(x)$ is an odd function

and

$\cos(-x) = \cos(x)$ is an even function.

If we adopt Boas' bound convention, then we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

We can introduce another expansion

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

With the cosine series we also need to introduce the average value of the function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

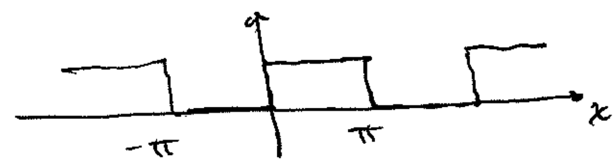
This leads to the total expansion

$$f(x) = \frac{a_0}{2} + a_1\cos(x) + a_2\cos(2x) + \cdots$$
$$+\, b_1\sin(x) + b_2\sin(2x) + \cdots$$

Note carefully that we define the expansion with the zero order term $a_0/2$. This is largely for convenience, that is, with this definition all of the a_n and a_0 are computed using the same formula.

Let's do an example of the combined series.

Ex:



$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

For

$$a_0 = \frac{1}{\pi}\int_0^{\pi} dx = 1$$

We have

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos(nx)\cdot 0\,dx + \int_0^{\pi}\cos(nx)\cdot(+1)\,dx\right.$$

$$= \frac{1}{\pi}\int_0^{\pi}\cos(nx)\,dx = \frac{1}{\pi}\left.\frac{\sin(nx)}{n}\right|_0^{\pi}$$

$$= 0 \quad \text{for } n \neq 0$$

Now,

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\sin(nx)\cdot 0\,dx + \int_0^{\pi}\sin(nx)\cdot 1\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin(nx)\,dx = -\left.\frac{\cos(nx)}{n\pi}\right|_0^{\pi}$$

$$= -\frac{1}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= -\frac{1}{n\pi}\left((-1)^n - 1\right)$$

Then

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd.} \end{cases}$$

Finally,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \cdots\right]. \blacksquare$$

Zak did a great job in his guest lecture and we all gave him constructive critiques.