Math Metods
Day 16

Today

Last Time

- We discussed the first exam at length. We reviewed the connection between the various vector products (scalar multiplication, dot product, and cross product) and the vector derivatives (gradient, divergence, and the newly introduced curl).
- Zak gave a guest lecture where he introduced the Fourier cosine series. He showed that in general a function is a sum of Fourier sine and cosine series.
- I want to build on and extend Zak's motivation and insight.

General Fourier Series

In the class before last we found a very unusual way to write the function $f(x) = 1$ on the interval $0 \le x \le L$. It was

$$1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + \cdots\right).$$

At the end of that class, and Zak noted this again, we realized that this series had an 'odd' behavior outside the interval $0 \le x \le L$. For example, from $-L \le x \le 0$ this series gives negative one. In retrospect this is quite natural since the sine function is an odd function, that is, $\sin(-x) = -\sin(x)$. Thus, one way of viewing Zak's result is to say that if we want a function that is one on the whole interval $-L \le x \le L$, or in Boas' convention $-l \le x \le l$ that we now adopt, we must include even functions in our Fourier series and the even analog of the sine function is the cosine function.

These observations lead to a nice insight, if the function you are trying to Fourier expand is purely odd, all you need is a Fourier sine series, or if it is purely even then all you need is a Fourier cosine series (along with Zak's constant term a_0). Strikingly, any function can be written as the sum of an even part and an odd part:

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{always even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{always odd}}.$$

So(!), any old Dirichlet function, neither purely even or odd, can be written as the sum of a sine series & a cosine series:

$$f(x) = \sum_{n=0}^{\infty}\left\{a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)\right\},$$

with the coefficients that Zak discussed:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx,$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

Notice that, completely consistently, $b_0 = 0$ always, and

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{0\cdot\pi}{l}x\right)dx = \frac{1}{l}\int_{-l}^{l} f(x)dx.$$

It is the nice internal consistency of these formulas that leads Boas to define a_0 such that the expansion of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + a_1\cos\left(\frac{\pi}{l}x\right) + \cdots + b_1\sin\left(\frac{\pi}{l}x\right) + \cdots.$$

The appearance of both sine and cosine functions in these general Fourier series suggeests that we could also try exponential Fourier series:

$$\cos\left(\frac{n\pi}{l}x\right) = \frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2},$$
$$\sin\left(\frac{n\pi}{l}x\right) = \frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}.$$

Putting this into our combined sine-cosine series gives

$$f(x) = \sum_{n=0}^{\infty}\left\{\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)e^{in\pi x/l} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right)e^{-in\pi x/l}\right\}$$
$$= \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/l}.$$

The second equality here encodes two nice insights: first, if we extend the lower bound of summation to negative values then many of the minus signs in the formula can be absorbed into the minus values of n, and second, that with this extension the coefficient can be written as a single complex constant, c_n.

The result is called an exponential Fourier series. You'll explore these more in Friday's discussion with Paul.

Solving the Wave Equation with Normal Modes

Recall our expression for the general solution to the wave equation:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi v}{L}t + \phi_n\right).$$

(Here we have returned to working with our string of total length L, so $0 \le x \le L$.) Let's reorganize this expression in a new way. By using a trigonometric expansion of the final term we can write the first and final terms as

$$A_n \cos\phi_n \cos\left(\frac{n\pi v}{L}t\right) - A_n \sin\phi_n \sin\left(\frac{n\pi v}{L}t\right).$$

It's then convenient to introduce the shorthands $C_n \equiv A_n \cos\phi_n$ and $D_n \equiv -A_n \sin\phi_n$. With these definitions we have

$$y(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{n\pi v}{L}t\right) + D_n \sin\left(\frac{n\pi v}{L}t\right)\right] \sin\left(\frac{n\pi}{L}x\right)$$

and hence

$$\dot{y}(x,t) = \sum_{n=1}^{\infty} \left[-C_n \sin\left(\frac{n\pi v}{L}t\right) + D_n \cos\left(\frac{n\pi v}{L}t\right)\right] \frac{n\pi v}{L} \sin\left(\frac{n\pi}{L}x\right).$$

In the case of the plucked string, we must then have

$$\dot{y}(x,0) = 0 = \sum_{n=1}^{\infty} \frac{n\pi v}{L} D_n \sin\left(\frac{n\pi}{L}x\right) \implies \boxed{D_n = 0}$$

and the general solution simplifies, becoming

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi v}{L}t\right) \sin\left(\frac{n\pi}{L}x\right).}$$

In the case of the hammered string, we similarly have

$$y(x,0) = 0 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) \implies \boxed{C_n = 0}$$

and the general solution again simplifies, this time becoming

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi v}{L}t\right) \sin\left(\frac{n\pi}{L}x\right).}$$

Having established these simplified forms of the general solution, let's return to our plucked string and find the general solution for a particular case in all the gory details.

Example: Let's focus on the plucked string at right. It has $\dot{y}(x,0) = 0$ and we will assume that it has been plucked to a height h at a distance of one quarter the string length along the string.

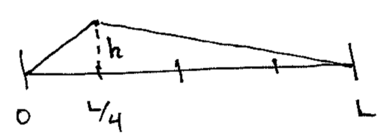

Figure 1: A specific example of the plucked string.

We want to impose this initial condition on our general solution for the plucked string, so we set

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) = \begin{cases} \frac{4h}{L}x, & 0 \le x \le \frac{L}{4} \\ -\frac{4h}{3L}x + \frac{4h}{3}, & \frac{L}{4} \le x \le L. \end{cases}$$

Here I have solved for the linear edges of the triangle much as we did for the integration problem on Day 10.

This is a Fourier sine series. So we can solve for the coefficients C_n using Fourier's trick:

$$C_n = \frac{2}{L}\left\{\int_0^{\frac{L}{4}} \frac{4h}{L} x \sin\left(\frac{n\pi}{L}x\right) dx + \int_{\frac{L}{4}}^{L} \left(-\frac{4h}{3L}x + \frac{4h}{3}\right) \sin\left(\frac{n\pi}{L}x\right) dx\right\}.$$

In the Day 14 notes we showed that

$$\int x \sin\left(\frac{n\pi}{L}x\right) dx = -\frac{Lx}{n\pi}\cos\left(\frac{n\pi}{L}x\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right),$$

which will help us to evaluate the integrals for C_n piece by piece. The first integral is

$$\begin{aligned}\frac{2}{L}\int_0^{\frac{L}{4}} \frac{4h}{L} x \sin\left(\frac{n\pi}{L}x\right) dx &= \frac{8h}{L^2}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi}{L}x\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\right]_0^{\frac{L}{4}} \\ &= \frac{8h}{L^2}\left[-\frac{L^2}{4\pi n}\cos\left(\frac{n\pi}{4}\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{4}\right)\right] \\ &= \frac{8h}{n\pi}\left[-\frac{1}{4}\cos\left(\frac{n\pi}{4}\right) + \frac{1}{n\pi}\sin\left(\frac{n\pi}{4}\right)\right].\end{aligned}$$

The second integral is

$$\begin{aligned}&-\frac{8h}{3L^2}\int_{\frac{L}{4}}^{L} x \sin\left(\frac{n\pi}{L}x\right) dx = -\frac{8h}{3L^2}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi}{L}x\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\right]_{\frac{L}{4}}^{L} \\ &= -\frac{8h}{3L^2}\left[-\frac{L^2}{n\pi}\cos(n\pi) + \frac{L^2}{(n\pi)^2}\sin(n\pi) + \frac{L^2}{4\pi n}\cos\left(\frac{n\pi}{4}\right) - \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{4}\right)\right] \\ &= -\frac{8h}{3\pi n}\left[-\cos(n\pi) + \frac{1}{4}\cos\left(\frac{n\pi}{4}\right) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{4}\right)\right].\end{aligned}$$

The third, and final, integral is

$$\frac{8h}{3L}\int_{\frac{L}{4}}^{L} \sin\left(\frac{n\pi}{L}x\right) dx = \frac{8h}{3L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{\frac{L}{4}}^{L} = \frac{8h}{3\pi n}\left[-\cos(n\pi) + \cos\left(\frac{n\pi}{4}\right)\right].$$

Adding these three integrals up, there is a remarkable cancelation and we are left with

$$C_n = \frac{24h}{3(n\pi)^2}\sin\left(\frac{n\pi}{4}\right) + \frac{8h}{3(n\pi)^2}\sin\left(\frac{n\pi}{4}\right) = \boxed{\frac{32h}{3(n\pi)^2}\sin\left(\frac{n\pi}{4}\right).}$$

This means that when we pluck a string as in this example, the complete, time-dependent solution to the wave equation can be written as

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \frac{32h}{3(n\pi)^2}\sin\left(\frac{n\pi}{4}\right)\cos\left(\frac{n\pi v}{L}t\right)\sin\left(\frac{n\pi}{L}x\right).}$$

This is a remarkable solution and well deserves the time that it would take to put together an animation showing how the wave evolves according to this solution. Even better the animation should show how each component evolves in time and then show what happens when you add these components up. Take up this challenge!