Today

Math Methods II

Day 3

I Last time

II 3D geometry with vectors

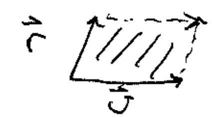
III Bivectors & the Wedge Product

I Using angular momentum as an example we reviewed the cross product:

(1) $\vec{l} = m(\vec{r}\times\vec{v}) \Rightarrow l = mrv\sin\theta$

$\hat{l}$ determined by right hand rule.

(2) l = area of ∥-ogram

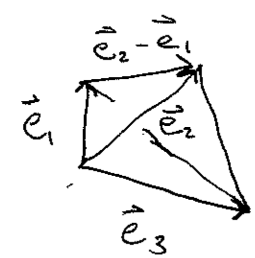


(3)

$$\vec{l} = m\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

Reviewed definitions of span, basis, dimension of a vector space, and orthonormal bases.

II In quantum gravity a classical model for a grain of space is a tetrahedron.



Suppose you are given 3 of its edge vectors. Construct the 4 vectors perpendicular to its faces...

and such that the magnitudes equal the face areas:

$$\vec{A}_1 = \tfrac{1}{2}\, \vec{e}_3 \times \vec{e}_2$$

$$\vec{A}_2 = \tfrac{1}{2}\, \vec{e}_1 \times \vec{e}_3$$

$$\vec{A}_3 = \tfrac{1}{2}\, \vec{e}_2 \times \vec{e}_1$$

$$\vec{A}_4 = \tfrac{1}{2}(\vec{e}_2 - \vec{e}_1) \times (\vec{e}_3 - \vec{e}_1)$$

$$= \tfrac{1}{2}\, \vec{e}_2 \times \vec{e}_3 - \tfrac{1}{2}\vec{e}_1 \times \vec{e}_3 - \tfrac{1}{2}\, \vec{e}_2 \times \vec{e}_1$$

$$= -\vec{A}_1 - \vec{A}_2 - \vec{A}_3$$

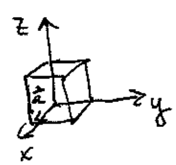
$$\sum_{i=1}^{N} \vec{F}_i = 0$$

But a pressure force is $\vec{F}_i = P \cdot \vec{A}_i$,

so

$$\sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} P \cdot \vec{A}_i = P \sum_{i=1}^{N} \vec{A}_i = 0$$

$$\Rightarrow \sum_{i=1}^{N} \vec{A}_i = 0.$$



Find the volume of a cube with side length L using vectors

or more symmetrically,

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = \vec{0}.$$

This is an example of Minkowski's theorem

$$\sum_{i=1}^{N} \vec{A}_i = 0 \quad \longleftrightarrow$$

convex polyhedron with N faces.

Physical proof of $\longleftarrow$ direction: Immerse the polyhedron in water. The pressure forces cancel out, so

Well, if we let $\vec{a} = L\hat{x}$, $\vec{b} = L\hat{y}$, and $\vec{c} = L\hat{z}$ then the area of the base is expressible as

$$\vec{a} \times \vec{b} = L^2\, \hat{x} \times \hat{y} = L^2\, \hat{z}$$

and dotting in $\vec{c}$ gives

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = L^3 = \mathrm{Vol}(\mathrm{Cube})$$

Note that

$$\vec{b} \times \vec{c} = L^2\, \hat{x}$$

$$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) \overset{\text{also}}{=} \vec{b} \cdot (\vec{c} \times \vec{a})$$

In fact shearing transformations preserve volume, so,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \text{Vol(parallelepiped)}$$

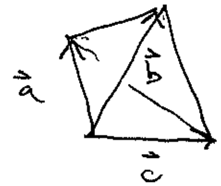
where $\vec{a}$, $\vec{b}$, $\vec{c}$ are the edge vectors with tails that meet at the corner of any parallelepiped.

There is an interesting relation to the volume of a tetrahedron here too. In highschool they may have taught you that for any pyramid

$$V = \frac{1}{3} A h$$

area of base ↑ ↑ height

So,



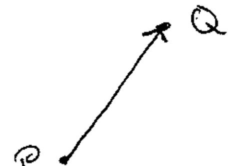
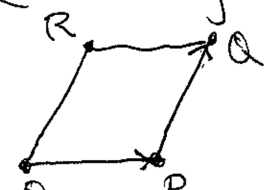
$$A = \frac{1}{2}|\vec{c}\times\vec{a}|$$

and

$$V = \frac{1}{6} h|\vec{c}\times\vec{a}| = \frac{1}{6}\,\vec{b}\cdot(\vec{c}\times\vec{a})$$

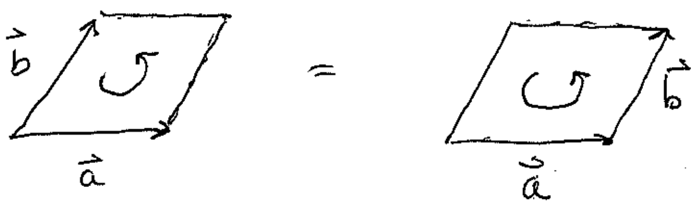
This suggests, and it is true, that any parallelepiped can be decomposed into 6 equal volume tetrahedra.

III Analogies are often useful when you meet new quantities — bivectors are like 2D generalizations of vectors

Vector (directed line segment)		Bivector (directed plane segment)
		
	Magnitude	
1. length of PQ		1. area of OPQR
	Direction	
2. From P to Q		2. Sense of rotation from O → P → Q Just curling of fingers on right hand

The simplest way to build a bivector is to take two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ and to form their <u>wedge</u> product (or exterior product)

$$\vec{a} \wedge \vec{b}$$



The wedge product is defined by two properties:

$$\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a} \quad \text{(anti-symm.)}$$

and linearity in both slots

$$(\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2) \wedge \vec{b}$$
$$= \lambda_1 \vec{a}_1 \wedge \vec{b} + \lambda_2 \vec{a}_2 \wedge \vec{b}$$

and

$$\vec{a} \wedge (\mu_1 \vec{b}_1 + \mu_2 \vec{b}_2)$$
$$= \vec{a} \wedge \mu_1 \vec{b}_1 + \vec{a} \wedge \mu_2 \vec{b}_2$$
$$= \mu_1 \vec{a} \wedge \vec{b}_1 + \mu_2 \vec{a} \wedge \vec{b}_2$$

That's it. All the other properties follow from these. For example,

$$\overbrace{\vec{a} \wedge \vec{a}}^{\text{switch 'em}} = -\vec{a} \wedge \vec{a},$$

but the only object that equals itself is zero, so

$$\vec{a} \wedge \vec{a} = 0. \leftarrow \text{zero bivector}$$

On the homework you will prove that the space of all bivectors in $\mathbb{R}^3$, called $\Lambda^2 \mathbb{R}^3$, is itself a vector space.

Next let's build a basis for this vector space. We can do this starting from a basis of $\mathbb{R}^3$, call it $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. Then we form wedge products

$$\{\hat{e}_1 \wedge \hat{e}_2,\ \hat{e}_1 \wedge \hat{e}_3,\ \hat{e}_2 \wedge \hat{e}_3\}$$

These are all of them, since

$$\hat{e}_2 \wedge \hat{e}_1 = -\hat{e}_1 \wedge \hat{e}_2 \quad \text{and} \quad \hat{e}_1 \wedge \hat{e}_1 = 0.$$

This means that

$$\dim \Lambda^2 \mathbb{R}^3 = 3.$$

This is the key to the idea of the cross product; we map these three bivectors back onto the basis vectors of $\mathbb{R}^3$

$$\begin{array}{ccccc} \{\hat{e}_1 \wedge \hat{e}_2, & & \hat{e}_1 \wedge \hat{e}_3, & & \hat{e}_2 \wedge \hat{e}_3\} \\ \downarrow & & \downarrow & & \downarrow \\ \{\hat{e}_3 = \hat{z}, & & -\hat{e}_2 = -\hat{y}, & & \hat{e}_1 = \hat{x}\}. \end{array}$$

The agreement of

$$\dim \Lambda^2 \mathbb{R}^3 = \dim \mathbb{R}^3$$

doesn't hold for other dimensions

$$\dim \Lambda^2 \mathbb{R}^n \neq \dim \mathbb{R}^n.$$