Math Methods
Day 4

Today

0. Regrades & Office Hours
I Last time
II Angular velocity, Rotation, & the BAC-CAB rule
III The derivative of a vector

I • Showed that the area vectors of a tetrahedron satisfy

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = \vec{0}$$

(Holds for all convex polyhedra).

• Found the volume of a parallelepiped spanned by $\vec{a}, \vec{b}, \vec{c}$:

$$V = \vec{a}\cdot(\vec{b}\times\vec{c}) = \text{cyclic permutations}$$
$$= -\vec{a}\cdot(\vec{c}\times\vec{b}) \text{ etc.}$$

[Also, $V_{tet} = \frac{1}{6}\vec{a}\cdot(\vec{b}\times\vec{c})$.]

• Found

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \det\begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}.$$

• Defined bivectors

Mag = area ‖-ogram
dir = sense of rotation

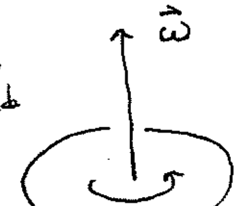
Wedge product
$\vec{a}\wedge\vec{b} = -\vec{b}\wedge\vec{a}$ and multilinear

Basis of $\Lambda^2\mathbb{R}^3$: $\{\hat{e}_1\wedge\hat{e}_2, \hat{e}_3\wedge\hat{e}_1, \hat{e}_2\wedge\hat{e}_3\}$.

II In physics we define the angular velocity vector using the right-hand rule (this should give away the fact that it's really a bivector)

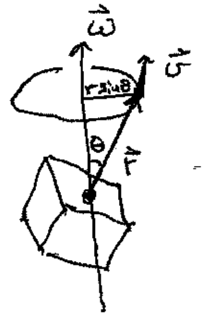
$[\omega] = \frac{\text{rad}}{s}$

Curl fingers of right hand with spin, thumb points along $\vec{\omega}$



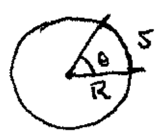
Record spinning as indicated

Now, suppose we have a rotating object, say a satallite with an antenna. How does the angular velocity of the tip of the antenna relate to its velocity $\vec{v}$?



Recall that arclength and angle are related by

$$S = R\theta$$



So,

$$R\omega = R\frac{d\theta}{dt} = \frac{d}{dt}(R\theta) = \frac{ds}{dt} = v$$

and

$$|\vec{\omega} \times \vec{r}| = \omega r \sin\theta = v$$

We have just shown that

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}.$$

Let's consider

$$\vec{\omega} \times \vec{r}.$$

The cross product vector points into the page at the moment illustrated and more generally is always tangent to the circle. Hence it is always parallel to $\vec{v}$. Furthermore

$$|\vec{\omega} \times \vec{r}| = \omega \underbrace{r\sin\theta}_{\text{radius of the circular trajectory of the tip.}}$$

Now we can ask after the relation between $\vec{l}$ and $\vec{\omega}$.

We have

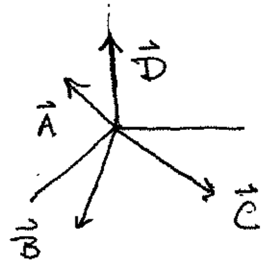
$$\vec{l} = m(\vec{r} \times \vec{v})$$
$$= m(\vec{r} \times (\vec{\omega} \times \vec{r})).$$

This result suggests we should study

$$\vec{A} \times (\underbrace{\vec{B} \times \vec{C}}_{\text{call this } \vec{D}})$$

carefully

First we have $\vec{D} = \vec{B} \times \vec{C}$ is perpendicular to the plane spanned by $\vec{B}$ and $\vec{C}$



Next $\vec{A}$ is general, but since

$$\vec{A} \times \vec{D}$$

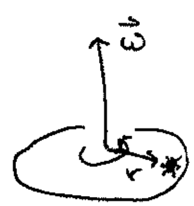
is perpendicular to $\vec{A}$ and $\vec{D}$ it must lie in the plane spanned by $\vec{B}$ and $\vec{C}$.

We'll call this the BAC-CAB rule.

Let's apply it to our angular momentum example

$$\vec{l} = m(\vec{r} \times (\vec{\omega} \times \vec{r})) = m\vec{\omega}(\vec{r}\cdot\vec{r}) - m\vec{r}(\vec{r}\cdot\vec{\omega})$$
$$= m r^2 \vec{\omega} - m\vec{r}(\vec{r}\cdot\vec{\omega})$$

Let's apply this to the case of a bug on a record



This means that

$$\vec{A} \times \vec{D} = c_1 \vec{B} + c_2 \vec{C}$$

with c_1 and c_2 real coefficients.

Deriving c_1 and c_2 takes algebra — we'll skip it and claim (you'll prove this on the homework)

$$c_1 = \vec{A}\cdot\vec{C} \quad \text{and} \quad c_2 = -\vec{A}\cdot\vec{B}.$$

Finally, then

$$\boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$$

In this case $\vec{r} \perp \vec{\omega} \Rightarrow \vec{r}\cdot\vec{\omega} = 0$ and

$$\vec{l} = m r^2 \vec{\omega}$$

a familiar result from intro physics. More generally the 2nd term will contribute.

III Define

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{x} + \frac{dA_y}{dt}\hat{y} + \frac{dA_z}{dt}\hat{z}$$

For example if $\vec{r}$ is a position vector, then

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

and

$$\vec{v} = \frac{d\vec{r}}{dt} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$
$$= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

and similarly

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}.$$

<u>Product Rules</u>: As an example, consider uniform circular motion for which,

$$\vec{r}\cdot\vec{r} = r^2 = \text{const.}$$

and

$$\vec{v}\cdot\vec{v} = v^2 = \text{const.}$$

Then

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d\vec{r}}{dt}\cdot\vec{r} + \vec{r}\cdot\frac{d\vec{r}}{dt}$$
$$= 2\vec{r}\cdot\frac{d\vec{r}}{dt} = \frac{d}{dt}(\text{const.}) = 0$$

$$\Rightarrow \vec{r}\cdot\vec{v} = 0 \Rightarrow \vec{r}\perp\vec{v}$$

as it should be. Quite similarly

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 2\vec{v}\cdot\vec{a} = 0 \Rightarrow \vec{v}\perp\vec{a},$$

which fits with the usual discussion of centripetal acceleration.

More generally

$$\frac{d}{dt}(a\vec{A}) = \frac{da}{dt}\vec{A} + a\frac{d\vec{A}}{dt}$$

[Pf sketch:

$$\frac{d}{dt}(a\vec{A}) = \frac{d}{dt}\left(aA_x\hat{x} + aA_y\hat{y} + aA_z\hat{z}\right)$$

apply product rule to each component]

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

and

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt},$$

notice that order matters in this case. All the proofs proceed similarly.

Next time: derivatives of varying basis vectors.