Math Methods
Day 5

Today

I Last time

II Derivatives of a variable basis, vector fieds, & integral curves

0. A computation that will help you on the homework: S'pose B is simple, i.e. $B = \vec{a} \wedge \vec{b}$ then

$$B \wedge B = \vec{a} \wedge \vec{b} \wedge \vec{a} \wedge \vec{b} = -\vec{a} \wedge \vec{a} \wedge \vec{b} \wedge \vec{b}$$
$$= \vec{a} \wedge \vec{a} \wedge \vec{b} \wedge \vec{b} \Rightarrow \boxed{B \wedge B = 0}$$

I • Reviewed $\vec{\omega}$ and

$$\vec{v} = \vec{\omega} \times \vec{r}$$

as well as

$$\vec{l} = m(\vec{r} \times (\vec{\omega} \times \vec{r}))$$

- Explored the BAC-CAB rule

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

- Defined

$$d\vec{A}/dt = \frac{dA_x}{dt}\hat{x} + \frac{dA_y}{dt}\hat{y} + \frac{dA_z}{dt}\hat{z}$$

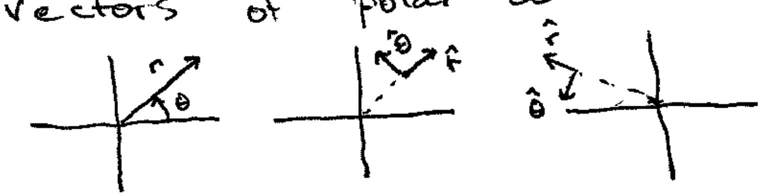
- Introd 3 product rules

$$\frac{d}{dt}(a\vec{A}) = \frac{da}{dt}\vec{A} + a\frac{d\vec{A}}{dt}$$

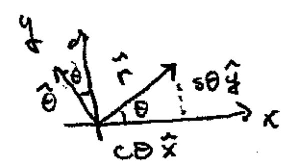
$$\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}.$$

- Briefly explored the varying basis vectors of polar coord.s:



II We agreed at the end of class that $\hat{r}$ and $\hat{\theta}$ were functions of θ, but not of r. What is their θ dependence?



So, $\hat{r} = c\theta\,\hat{x} + s\theta\,\hat{y}$

One way to get $\hat{\theta}$ is to compute

$$\hat{\theta} = \hat{z} \times \hat{r} = -s\theta\,\hat{x} + c\theta\,\hat{y}.$$

Now the vector derivative is straightforward

$$\frac{d\hat{r}}{dt} = -s\theta \frac{d\theta}{dt}\hat{x} + c\theta \frac{d\theta}{dt}\hat{y}$$

$$= \frac{d\theta}{dt}\left(-s\theta\hat{x} + c\theta\hat{y}\right) = \frac{d\theta}{dt}\hat{\theta}$$

and

$$\frac{d\hat{\theta}}{dt} = -c\theta\frac{d\theta}{dt}\hat{x} - s\theta\frac{d\theta}{dt}\hat{y}$$

$$= -\frac{d\theta}{dt}\left(c\theta\hat{x} + s\theta\hat{y}\right) = -\frac{d\theta}{dt}\hat{r}$$

An interesting application of all of this is $\vec{r} = r\hat{r}$, then

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= v_r\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$$

$$= v_r\hat{r} + \underbrace{r\omega}_{\text{angular speed}}\hat{\theta}$$

We can also take

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\omega\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} + r\omega\frac{d\hat{\theta}}{dt}$$

$$\Rightarrow \vec{a} = (a_r - r\omega^2)\hat{r} + (r\alpha + 2v_r\omega)\hat{\theta}$$

It's pretty neat to see all the acceleration terms come out directly from a computation.

Fields: The concept of a field is extremely useful in physics. The simplest example is a scalar field, which is just another name for a function, e.g. the temperature in this room. Feed in a point of the room and the field returns the temperature $T(x,y,z)$ at that point. A richer example is a vector field, which returns a vector for every point in a region. You can think of this as a vector-valued function. For example

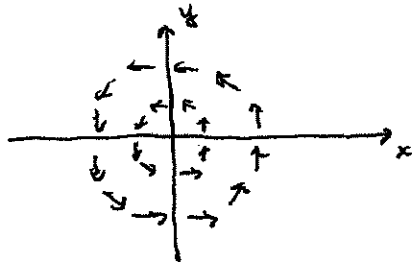
- $\hat{r} = \hat{r}(r,\theta) = \hat{r}(\theta) = c\theta\hat{x} + s\theta\hat{y}$

  it returns the $\hat{r}$ vector for each point (r,θ) of the polar plane.

- $\vec{E}(x,y,z)$

  returns a vector pointing along the electric field at each pt. (x,y,z).

We sketch vector fields as a collection of arrows:



Note that there is an arrow defined at every pt., but we don't draw them all because the picture would be solid black. You've almost certainly

from the vector field: if a particle moves along a curve what vector is always tangent to that curve? velocity! If you know the velocity how do you recover the curve?

$$\frac{d\vec{r}}{dt} = \vec{v} \quad \leftarrow \text{known}$$

$$\Rightarrow \quad \vec{r} = \int \vec{v}\, dt$$

This gives the "integral curves" of

heard the term electric field before, but you may have associated it with electric field lines not with the field of arrows. What gives? In fact there is a direct relation between a vector field and its field lines. At every point the vector field is tangent to the field line — this actually suggests a way to find the field lines

the vector field, or as the physicists would call them the field lines. Example:

Consider again our rotating record

$\frac{d\theta}{dt} = \omega = \text{const.}$

$$\frac{d\vec{r}}{dt} = \vec{v} = r\omega\,\hat{\theta} = r\omega(-s\theta\,\hat{x} + c\theta\,\hat{y})$$
$$= r\omega(-s\omega t\,\hat{x} + c\omega t\,\hat{y})$$

$$\Rightarrow \vec{r} = \int \vec{v}\, dt = -r\omega \int s\omega t\, dt\, \hat{x} + r\omega \int c\omega t\, dt\, \hat{y}$$

So,

$$\vec{r} = r\omega \frac{1}{\omega} c\omega t\, \hat{x} + r\omega \frac{1}{\omega} s\omega t\, \hat{y}$$

$$= r\, c\omega t\, \hat{x} + r\, s\omega t\, \hat{y}$$

What curve is this? A circle(!),

e.g.

$$\vec{r}\cdot\vec{r} = r^2 c^2\omega t + r^2 s^2\omega t = r^2 = \text{const.}$$

Geometrically it is also clear that the circle provides a curve that is tangent to all the vectors of this field.