Math Methods
Day 6

Today

I Last time

II More examples of integral curves

III Vector derivatives I: The Gradient

I • Found

$$\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta}, \quad \frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{r}$$

• Intro'd vector fields, e.g. $\vec{E}(x,y,z)$ or $\vec{v}(x,y,z)$

• Found the integral curve of a velocity vector field using

$$\frac{d\vec{r}}{dt} = \vec{v} \Rightarrow \vec{r} = \int \vec{v}\, dt.$$

II But, more generally, what defines an integral curve?

First important point: an integral curve need not be parametrized by time. We don't usually think of an electric field line as being traversed in time. To emphasize this point we will often take the parameter of an integral curve to be λ, which has no interpretational overlay.

Def: An integral curve of a vector field is a curve whose tangent vectors agree with the given vector field. If $\vec{u}$ is the given vector field and $\vec{r}$ is the position on the integral curve parametrized by λ, this can be written

$$\frac{d\vec{r}}{d\lambda} = \vec{u}$$

and finding the integral curve amounts to solving this Eq. for $\vec{r}(\lambda)$.

Let's do some examples. Coordinates actually furnish a nice example.

<u>Ex</u>. Find the coordinate curves of the Cartesian basis $\{\hat{x}, \hat{y}\}$ of $\mathbb{R}^2$.

The basis vector $\hat{x}$ is a vector field. It doesn't depend on position. The integral curve condition is (recall $\vec{r} = x\hat{x} + y\hat{y}$)

in this case

$$\frac{d\vec{r}}{d\lambda} = \vec{u} = \hat{x}$$

or $$\frac{dx}{d\lambda}\hat{x} + \frac{dy}{d\lambda}\hat{y} = \hat{x}$$

Another nice example is the electric field of a point charge:

$$\vec{E} = k\frac{q}{r^2}\hat{r}, \quad \text{with } k \equiv \frac{1}{4\pi\epsilon_0}$$

Here it is convenient to work in spherical polar coordinates (r, θ, ϕ). Again we have

$$\frac{d\vec{r}}{dt} = \vec{E}$$

and by symmetry the integral curves must have $\theta = \text{const.} = c_1$ $\phi = \text{const.} = c_2$

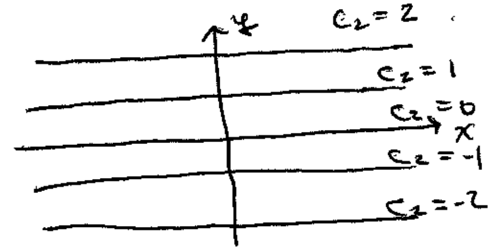
Working with the component equations, this is

$$\frac{dx}{d\lambda} = 1 \quad \text{and} \quad \frac{dy}{d\lambda} = 0$$

or

$$x = \lambda + c_1 \quad \text{and} \quad y = c_2$$

Choosing $\lambda = 0$ at $x = 0$ sets $c_1 = 0$ and for different values of c_2 we get the curves:



A similar treatment of $\hat{y}$ gives the vertical coord. curves.

So, this is just, $\vec{r} = r\hat{r}$ and

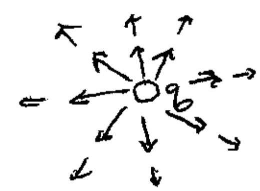
$$\frac{dr}{d\lambda} = k\frac{q}{r^2} \Rightarrow r^2\,dr = kq\,d\lambda$$

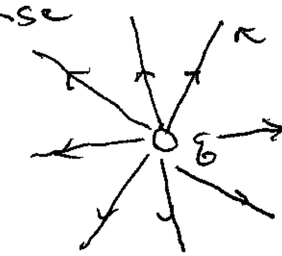
Integrating gives

$$r^3/3 = kq\lambda + \text{Const}$$

or $$\boxed{r(\lambda) = (3kq\lambda + c_0)^{1/3}}$$

This makes sense





$\lambda^{1/3}$ grows more slowly out here in agreement with the tangent vector getting smaller

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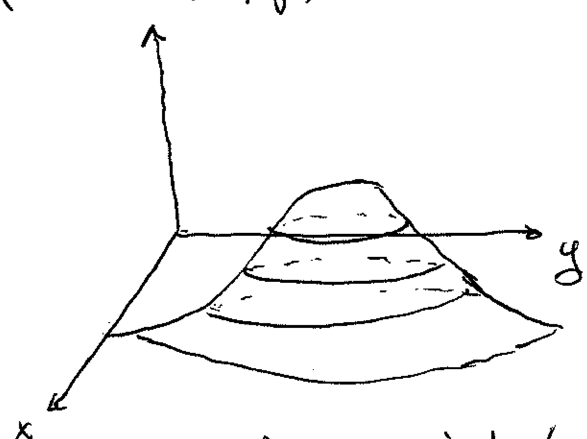
III We're starting to appreciate how useful it is to take derivatives and integrals of vector fields.

I don't think you'll be surprised that there are more complicated differential and integral operations for vector fields. The simplest of these operators is the <u>gradient</u>.

Consider a scalar field $f(x,y)$,

Say the height of a mountain or the temperature of a metal sheet $f(x,y)$



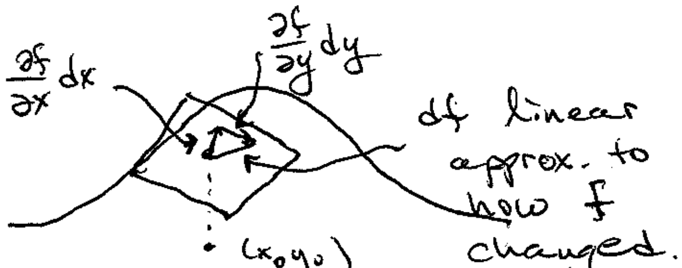
If we start at a point (x_0, y_0) of the xy-plane how does the height vary as we move around the plane?

Well it depends. It depends on what direction we move in. In some directions the height doesn't change and in others we would have to climb steeply. The total derivative or differential of f captures this

Mathematically:

$$df(x,y) = \underbrace{\frac{\partial f}{\partial x} dx}_{\text{height we have to climb if we move a little in the x-direction}} + \underbrace{\frac{\partial f}{\partial y} dy}_{\text{ditto for y-dir}}$$

Say like this



In the limit of smaller and smaller displacements, this becomes exact.

Notice that we can write

$$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (dx, dy)$$
$$\equiv \vec{\nabla} f \cdot d\vec{r}$$

The differential gives us how much we have to climb if we move in an arbitrary direction (a little in x, dx, <u>and</u> a little in y, dy). Why is it enough to just add these contributions?

It's because we're imagining the displacements are small enough that the hill can be approximated by a plane at (x_0, y_0).

Here

$$\vec{\nabla} f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

is the <u>gradient</u> of f.

Now consider a level set of f, that is, the set of all pts (x,y) s.t.

$$f(x,y) = \text{const.}$$

level set of f = set of pts of const. height in this example.

What if we chose to move along a $d\vec{r}$ such that we remain on this level set? Then $df = d(\text{const.}) = 0$ and hence

when $d\vec{r}$ along level set.

$$\vec{\nabla} f \cdot d\vec{r} = 0.$$

Geometrically what does this imply?

That $\vec{\nabla} f$ is perpendicular to the level sets of f!