Math Methods
Day 7

Today

I Last time

II The gradient $\vec{\nabla} f$ points in the direction of steepest increase of f.

III Definitions of the $\vec{\nabla}$ operator and the divergence $\vec{\nabla}\cdot\vec{V}$

I • Worked with integral curves. Recall that given a vector field $\vec{v}$, its integral curves are $\vec{r}(\lambda)$ such that the tangent vector $d\vec{r}/d\lambda$ agrees with $\vec{v}$. That is, we solve the differential equation

$$\frac{d\vec{r}}{d\lambda} = \vec{v}$$

for the curve $\vec{r}(\lambda)$. We did this for the coordinate curves of the $\hat{x}$ basis vector field and for $\vec{E}(r)$, the electric field of a point charge.

• We also studied the total derivative of a function $f(x,y)$:

$$df = \vec{\nabla} f \cdot d\vec{r} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

• We also proved that the gradient

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

is always perpendicular to the level sets $f(x,y,z) = \text{const}$.

Let's pick up at this point and find in which direction $\vec{\nabla} f$ points.

Recall:

What if we chose to move along a $d\vec{r}$ such that we remain on this level set? Then $df = d(\text{const.}) = 0$ and hence

$$\vec{\nabla} f \cdot d\vec{r} = 0.$$ ← when $d\vec{r}$ along level set.

Geometrically what does this imply?

II That $\vec{\nabla} f$ is perpendicular to the level sets of f! Ok, but there are two directions perp. to a level set, uphill and downhill, which one does gradient points uphill. Not just uphill, but along the steepest slope of the hill where df has its maximum value! ← read the change in height!

A second example, also easy to compute, is

$$f(x,y,z) = x^2+y^2+z^2,$$

The level sets of this function are $f = \text{const.} \Rightarrow x^2+y^2+z^2 = \text{const.}$,

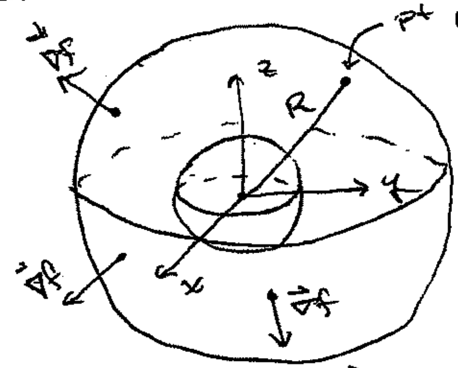
$\vec{\nabla} f$ point in? Let's consider $d\vec{r}$ in an arbitrary direction

$$d\vec{r} = \epsilon \hat{u}$$ ← a unit vector; $\epsilon \ll 1$ ← makes dr small

Then

$$df = \vec{\nabla} f \cdot d\vec{r} = \vec{\nabla} f \cdot (\epsilon \hat{u})$$
$$= \epsilon \vec{\nabla} f \cdot \vec{u}$$
$$= \epsilon |\vec{\nabla} f| \cos\theta$$

Maximized when $\cos\theta = 1$, hence when $\hat{u}$ points along $\vec{\nabla} f$. The which is the equation for a sphere in 3D



The gradient is $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
$$= (2x, 2y, 2z)$$
$$= 2(x,y,z)$$
This is the radial vector at every point.

Is $\vec{\nabla} f$ a vector field? Absolutely, it gives you a vector at every point

$$\vec{\nabla} f(x, y, z).$$

But this means it has integral curves.

These are the curves of "steepest ascent"; they're good for hard core hikers.

As such, it can act on a vector field $\vec{V}(x,y,z) = V_x(x,y,z)\hat{x} + V_y(x,y,z)\hat{y} + V_z(x,y,z)\hat{z}$ through the dot product

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

We call the result (a scalar) the divergence of $\vec{V}$.

III Today just definitions, next time we'll explore the physics. Define

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$= \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

We put the unit vectors to the left to make sure its clear that $\partial/\partial x$ is not acting on $\hat{x}$.

This is a vector of derivative operators.