Math Methods
Day 8

<u>Today</u>

I Last time

II Physical meaning of the divergence.

I
- Studied the total derivative of a function $f(x,y)$:

$$df = \vec{\nabla} f \cdot d\vec{r} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- Prove that the gradient

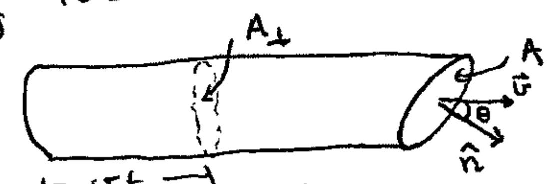
$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

is always perpendicular to the level sets $f(x,y,z) = \text{const.}$

- Further proved that $\vec{\nabla} f$ points in the direction of steepest increase of the function.
- The integral curves of the vector field $-\vec{\nabla} f$ are the curves of steepest descent.
- Defined the <u>divergence</u> of a vector field $\vec{V}(x,y,z)$ as

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$

II As a physical example, let's study the fluid flow down a pipe



Let $\vec{v}$ be the velocity vector field of the fluid, say water. The mass of water that flows by $A_\perp$ in time t is

$$\underset{\text{density of } H_2O}{\rho}\,\underbrace{(A_\perp)(vt)}_{\text{volume of } H_2O}$$

In the same time, the same water crosses the tilted surface A. More precisely, the flow perpendicular to this surface is

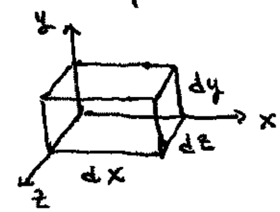
$$\rho (A_\perp)(v t) = \rho (A \cos\theta \, v)(t)$$

Then the mass flow per unit area, per unit time is

$$\rho v \cos\theta = \vec{V} \cdot \hat{n}$$

where $\vec{V} = \rho \vec{v}$ = mass flow per area per time.

Consider a small volume



The rate at which it flows out at right is

$$(\vec{V} \cdot \hat{x})\Big|_{x=dx} dy\,dz = V_x\Big|_{x=dx} dy\,dz$$

Then in this simple example the net outflow is

$$V_x\Big|_{x=dx} dy\,dz - V_x\Big|_{x=0} dy\,dz$$

$$= \left[V_x\Big|_{x=dx} - V_x\Big|_{x=0} \right] dy\,dz$$

$$= \left[\frac{\partial V_x}{\partial x} dx \right] dy\,dz$$

In a moment we will consider fluid flow out through each side, but for now just think of fluid flowing along the x-axis.

The rate at which fluid flows into the small region at the left is

$$(\vec{V} \cdot \hat{x})\Big|_{x=0} dy\,dz$$

evaluated at $x=0$; $dy\,dz$ ← area of surface

$$= V_x\Big|_{x=0} dy\,dz$$

The same analysis applies in the y- and z-directions, which have outflows

$\frac{\partial V_y}{\partial y} dx\,dy\,dz$ (through top / bottom)

$\frac{\partial V_z}{\partial z} dx\,dy\,dz$ (through front/back)

Then the <u>total</u> outflow is

$$\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx\,dy\,dz = \vec{\nabla} \cdot \vec{V} \, dx\,dy\,dz$$

Dividing by the volume of our region

$$\boxed{\vec{\nabla} \cdot \vec{V}\Big|_{pt.} = \text{net outflow per unit volume.}}$$

Fluid flow is one way for fluid to enter or leave a region, but another possibility is that there is a source or sink in the region, e.g. a little hose adding or sucking water out of the region. Call

ψ = source density − sink density
= net mass of fluid being added per unit time per unit volume.

Then the net rate of mass increase in a region $dx\,dy\,dz$ is

$$\frac{\partial \rho}{\partial t}\,dx\,dy\,dz = \psi\,dx\,dy\,dz - \vec{\nabla}\cdot\vec{V}\,dx\,dy\,dz$$

or canceling the volume

$$\frac{\partial \rho}{\partial t} = \psi - \vec{\nabla}\cdot\vec{V}$$

If nothing is adding or removing mass, $\psi = 0$, this simplifies to

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{V} = 0}$$

The continuity equation.

So, much as $\vec{\nabla} f$ has a differential equation associated with it, that of the integral curve, the divergence also has a differential eqn. associated to it – the continuity equation. When this equation is satisfied it tells you that you are not creating or destroying the "stuff" you are studying, just moving it around. A clear example is

$$\vec{\nabla}\cdot\vec{B} = 0$$

This says that there are no sources of magnetic flux, that is, no magnetic monopoles to create $\vec{B}$ field leaving a small region. Contrast

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

charge density