Math Methods
Day 9

Today

I Last time

II Comments on Multiple Integrals

III Change of Variables:
Geometry & Jacobians

I • Defined the divergence

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

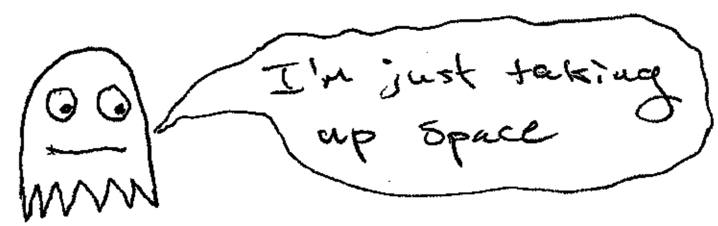
of a vector field $\vec{V} = \vec{V}(x,y,z)$.

• Found that this can be interpretted as

$$\vec{\nabla}\cdot\vec{V}|_{pt} = \text{net outflow per unit volume}$$

or

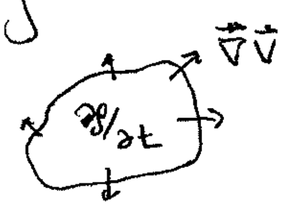
$$\vec{\nabla}\cdot\vec{V}|_{pt} = \text{net outward flux per Vol.}$$



• Derived the continuity eqn:

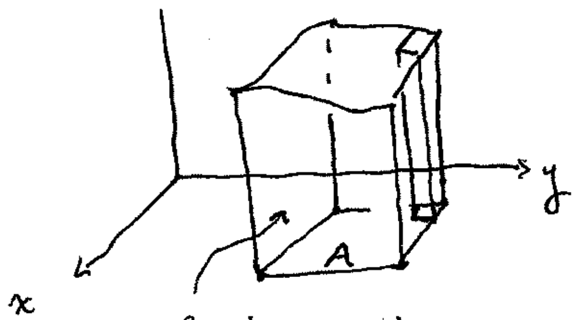
$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{V} = 0$$

When satisfied this eqn tells you that you are not creating or destroying the "stuff", just flowing it around.



II Much like in your first introduction to integration, you can think of a double integral as the volume of the region trapped between the surface $f(x,y)$

and the xy-plane below



Vol of this region

$$= \iint_A f(x,y)\,dx\,dy$$

When A is rectangular, things are pretty simple and it doesn't matter

you would want to do the y integral first! This is because the boundary of the region depends on what x you are at.

<u>Ex</u> Evaluate

$$\int_{x=0}^{2}\int_{y=x}^{2} e^{-y^2/2}\,dy\,dx.$$

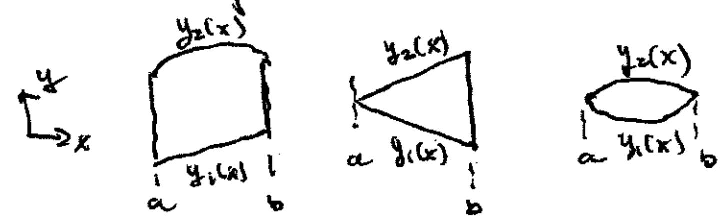
Note: this is difficult in the integral's present form. Figure out

Which order we do the integrals in:

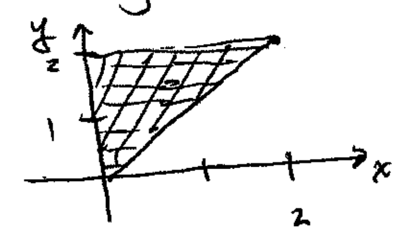
$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$

$$= \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy$$

But, in many other cases it does. For example if A has these shapes:



how to change the order of integration. First draw the region of integration



In the problem statement they are integrating over y first. To integrate over the same region, but doing the x integration first we need to integrand from $x=0$ to $x=y$ and from $y=0$ to $y=2$. Then

$$\int_{x=0}^{2}\int_{y=x}^{2} e^{-y^2/2}\,dy\,dx = \int_{y=0}^{2}\int_{x=0}^{y} e^{-y^2/2}\,dx\,dy$$

$$= \int_{y=0}^{2} e^{-y^2/2}\left(x\Big|_0^y\right)dy$$

$$= \int_{y=0}^{2} y\,e^{-y^2/2}\,dy$$

$$= -e^{-y^2/2}\Big|_0^2 = -e^{-2}+1 = \boxed{1-\frac{1}{e^2}}.$$

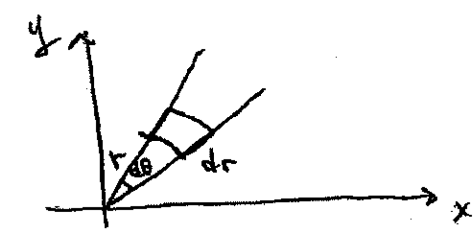
integrand for the 2nd integral.

Another way to change integrands is by transforming to a new coord. system. Today we will do this geometrically, next time we will find a convenient algebraic/calculus method to do the same.

III Consider polar coordinates for the plane, (r,θ). A small square swept out by going from r to $r+dr$ and from θ to $\theta+d\theta$

This shows that it is a very good idea to keep order of integration in mind. We've gone from an integral that can only expressed in terms of a special function to one that we were easily able to evaluate immediately.

Notice that the technical reason that this mattered was that switching the order changed the



has side lengths dr and $r\,d\theta$ (since the arclength of a circle of radius r is $s=r\theta$, so $ds=r\,d\theta$). The area of this square is

$$dx\,dy = r\,dr\,d\theta.$$

The factor r connecting these two descriptions is called the Jacobian of the transf.

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For a region A

$$\text{Area}(A) = \iint_A dx\,dy = \iint_A r\,dr\,d\theta$$

We did this geometrically. Now, we turn to other methods.

When the coordinate transformation between two systems is know there is a convenient way to compute the factor that connects their differential areas:

$$dx\,dy = |J|\,dr\,d\theta$$

The ~~factor~~ J is called the Jacobian of the coord. transform and is given by

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} \equiv \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

means a det.

Let's do it

$$x = r\cos\theta \qquad y = r\sin\theta$$

so,

$$\frac{\partial x}{\partial r} = c\theta \qquad \frac{\partial x}{\partial \theta} = -rs\theta$$

$$\frac{\partial y}{\partial r} = s\theta \qquad \frac{\partial y}{\partial \theta} = rc\theta$$

and

$$J = \begin{vmatrix} c\theta & -rs\theta \\ s\theta & rc\theta \end{vmatrix} = rc^2\theta + rs^2\theta$$
$$= r$$

that is, we specify $x(s,t)$ and $y(s,t)$. Then

$$dx = \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt$$

and

$$dy = \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt.$$

This then means that

$$dx \wedge dy = \left(\frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt\right) \wedge \left(\frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt\right)$$

and we recover

$$dx\,dy = r\,dr\,d\theta.$$

Where did J come from? We know that $\vec{a} \wedge \vec{b}$ spans the oriented area bounded by $\vec{a}$ & $\vec{b}$. Similarly, $dx \wedge dy$ spans a little differential oriented area. Now, suppose we make a general coord transformation from x, y to s, t,

You've already evaluated this wedge, it's

$$dx \wedge dy = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} ds \wedge dt$$

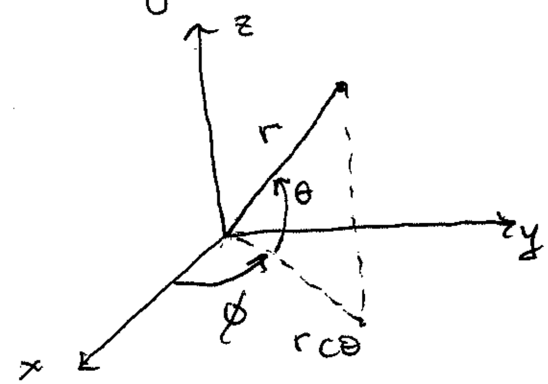
$$\equiv J\, ds \wedge dt$$

If we choose to "forget" the wedge orientation and keep track of this in our bounds of integration then

$$dx\,dy = |J|\, ds\,dt\ !$$

To solidify this, let's repeat the process in 3D Spherical coord.s. Let's do the Jacobian first

$$x = r s\theta c\phi, \quad y = r s\theta s\phi \quad z = r c\theta$$



Then

$$J = \begin{vmatrix} s\theta c\phi & r c\theta c\phi & -r s\theta s\phi \\ s\theta s\phi & r c\theta s\phi & r s\theta c\phi \\ c\theta & -r s\theta & 0 \end{vmatrix}$$

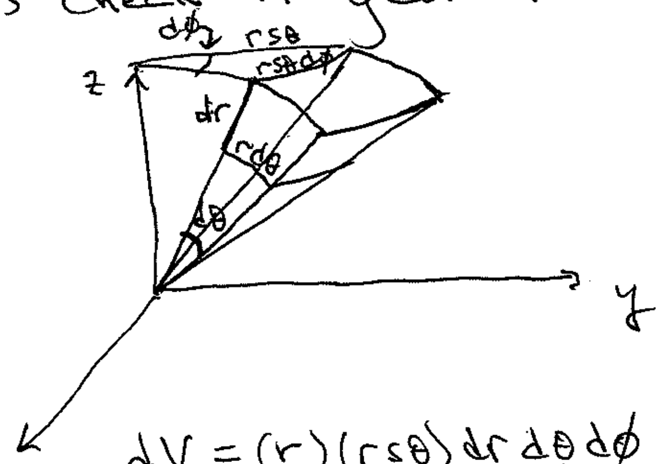
$$= s\theta c\phi \left(r^2 s^2\theta c\phi \right) - s\theta s\phi \left(-r^2 s^2\theta s\phi \right)$$
$$+ c\theta \left(r^2 c\theta s\theta c^2\phi + r^2 c\theta s\theta s^2\phi \right)$$

$$= r^2 s^3\theta + r^2 c^2\theta s\theta$$
$$= r^2 s\theta \left(s^2\theta + c^2\theta \right) = r^2 s\theta$$

Apparently then

$$dV = r^2 s\theta \, dr \, d\theta \, d\phi$$

Let's check it geometrically,



$$dV = (r)(r s\theta) \, dr \, d\theta \, d\phi$$
$$= r^2 s\theta \, dr \, d\theta \, d\phi.$$

In perfect agreement with the Jacobian calculation. The Jacobian technique can be very useful when you don't have geometrical insight into a coordinate transformation. An example of this is the parabolic coordinates you derived in the last problem of this week's homework. On the next homework you will derive the Jacobian for this coordinate system.