# Homework 10

Due April 21st, 2017 at 5pm

Reading Hecht Ch. 7, §§1-4 and Ch. 11, §§1-2.

1. In a *linear* and *homogeneous* (meaning that $\mu$ and $\epsilon$ don't vary from point to point) dielectric medium, the Maxwell equations are largely unaffected. In the absence of a current $\vec{J}$, the only equation that is modified is the Maxwell-Ampère law

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t},$$

where $\mu$ and $\epsilon$ are the magnetic permeability and permittivity in the dielectric material, instead of the values $\mu_0$ and $\epsilon_0$ in vacuum. Following through your previous derivation of the electromagnetic wave equation gives $v = 1/\sqrt{\epsilon\mu} = c/n$, where

$$n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

is the index of refraction of the material. For most materials, $\mu$ is very close to $\mu_0$ and the index of refraction reduces to

$$n \approx \sqrt{\frac{\epsilon}{\epsilon_0}} \equiv \sqrt{\epsilon_r},$$

where $\epsilon_r$ is the dielectric constant. For most materials the dielectric constant is greater than 1 and the speed of light in the medium is less than in vacuum.

Now, relax the assumption that the material is homogeneous and allow $\epsilon$ to vary from point to point. (**a**) Retrace your derivation of the wave equation and show that when $\epsilon$ is variable the wave equation becomes

$$\nabla^2 \vec{E} + 2\vec{\nabla}\left(\vec{E} \cdot \vec{\nabla} \ln[n]\right) - \frac{n^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0.$$

There is one more piece of background in electricity and magnetism that is useful for you to solve this problem. In the vacuum we had the relation $\vec{\nabla} \cdot \vec{E} = 0$, but in a material this is no longer true. This is because there are charges all over the medium. However, a closely related statement is true, namely, if we define the electric *displacement* $\vec{D} = \epsilon\vec{E}$, then

$$\vec{\nabla} \cdot \vec{D} = \rho_f,$$

where $\rho_f$ is the free charge density and captures the density of charge carriers that are free to run around the material. For this problem, I want you to go ahead and take $\rho_f = 0$.

(**b**) The Kirchhoff diffraction theory that we've been studying builds on the scalar wave equation. In the vacuum Maxwell equations we could look at each component of the electric and magnetic fields separately, thus arriving at six scalar wave equations. Is it possible to make this separation for the wave equation you found in part (**a**)? Explain your argument in mathematical terms.

2. Recall from class the divergence theorem

$$\int_{\text{Vol}V} \vec{\nabla} \cdot \vec{W} dV = \oint_{\text{Surf } S=\partial V} \vec{W} \cdot d\vec{S}.$$

Below you will use this to prove Green's first and second identities. (**a**) By using the divergence theorem with $\vec{W} = \phi \vec{\nabla} \psi$ prove that

$$\int_{\text{Vol } V} \left( \phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi \right) dV = \oint_{\text{Surf } S} \left( \phi \vec{\nabla} \psi \right) \cdot \hat{n} \, dS,$$

where $S$ is the boundary of the volume $V$.

(**b**) To prove the second identity, take the first one as is, then take a second copy with $\phi$ and $\psi$ interchanged; then take the difference of the two equations to get

$$\int_{\text{Vol } V} \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) dV = \oint_{\text{Surf } S} \left( \phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi \right) \cdot \hat{n} \, dS.$$

3. (**a**) Letting $\phi = U_1$ and $\psi = U_2$ with $U_1$ and $U_2$ solutions of the Helmholtz equation, use one of the Green's identities to prove

$$\oint_{\text{Surf } S} \left( U_1 \vec{\nabla} U_2 - U_2 \vec{\nabla} U_1 \right) \cdot \hat{n} \, dS = 0,$$

as we claimed in class.

(**b**) We also claimed that $U_2 = e^{ikr}/r$ was a solution of the Helmholtz equation. Confirm that this is true.

4. Hecht 7.20

5. (**a**) Hecht 7.29 & (**b**) Hecht 7.30 & 7.31