Notes on Complex Numbers in Physics

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The Algebra of Complex Numbers

Mathematically, complex numbers are introduced as solutions to algebraic equations. For example, quadratic equations $ax^2 + bx + c = 0$ have two solutions specified by the quadratic formula $x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$, but in many cases these solutions will involve square roots of negative numbers. In the simplest case, $x^2 + 1 = 0$, the solutions are $x = \pm\sqrt{-1}$. These square roots are not real numbers, and so are given a new designation, the **imaginary unit** $i \equiv \sqrt{-1}$, and its negative $-i \equiv -\sqrt{-1}$. With these definitions the following equations hold:

$$ii = i^2 = -1, \quad (-i)(-i) = i^2 = -1, \text{ and } (-i)i = i(-i) = 1$$

The imaginary unit can be multiplied by any real scalar, e.g. $3 * i = 3i$ or $\frac{-5}{2} * i = \frac{-5i}{2}$. Any number of the form bi, with b a real number, is known as an **imaginary number**. Imaginary numbers combine under addition and subtraction similarly to their real counterparts:

$$i + i = 2i, \qquad i - i = 0, \qquad \frac{-5i}{2} + \frac{3i}{4} = \frac{-7i}{4}.$$

While sums and differences of imaginary numbers are equal to another imaginary number, this is not the case if one adds or subtracts a real number and an imaginary number. So if a is a real number and bi and imaginary number, their sum $a + bi$ cannot be simplified and (unless either a or b is 0) is not equal to a real number or an imaginary number. Numbers of the form $z = a + bi$, where a and b are real numbers are known as **complex numbers**.[1] Real numbers and imaginary numbers are each subsets of complex numbers, the former being all complex numbers with $b = 0$, and the latter being all complex numbers with $a = 0$. The **real part** of a complex number $z = a + bi$ is a, a relation often written Re $[z] = a$, and the **imaginary part** is b, Im $[z] = b$. Note that the imaginary part is, by convention, the real coefficient and not the imaginary piece of the sum.

When adding two complex numbers $z = a + bi$ and $w = c + di$, the real and imaginary pieces of each number combine,

$$z + w = a + bi + c + di = (a + c) + (b + d)i,$$

forming a new complex number.

When multiplying two complex numbers, the multiplication rules of the imaginary unit are invoked:

$$z * w = (a + bi)(c + di) = ac + bci + adi + bdii = ac + bci + adi - bd = (ac - bd) + (bc + ad)i.$$

So the product of two complex numbers also yields a complex number. Similarly it is true that differences, quotients, roots, and other algebraic combinations of complex numbers yield complex numbers (except for division by 0). In mathematical parlance, the set of complex is said to be "algebraically closed," something which is not true of integers, rational numbers, or real numbers. This fact underlies the **fundamental theorem of algebra**, which states that single-variable algebraic equations of order n, $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$, where the a_n's are complex numbers, has n complex-number solutions.[2]

Physical Problems and the Complex Plane

Just because a number is a solution to an equation that models a physical question does not mean it possesses any physical meaning. Consider the problem of the length of the side of a square plot of land with an area of 900 m^2. The equation $x^2 = 900$ m^2 has two solutions, $x = 30$ m and $x = -30$ m. However, there being no physical concept corresponding to negative length, one disregards the second solution. A slightly more subtle problem is one where a ball, initially at height x_o and traveling upwards at speed v_o, eventually falls back to Earth due to gravitational acceleration g. In order to find the time when it hits the ground, one uses the constant acceleration equation $x = x_o + v_o t - \frac{1}{2} g t^2$ and sets $x = 0$. This yields the two solutions $t = \frac{1}{g}\left(v_o \pm \sqrt{v_o{}^2 + 2 g x_o}\right)$. The positive sign solution gives the desired time to impact, but the negative sign solution is not without meaning. If the ball was under constant gravitational acceleration before the moment in question, then it was also at ground level at $t = \frac{1}{g}\left(v_o - \sqrt{v_o{}^2 + 2 g x_o}\right)$, a time *before* this initial moment. Negative numbers can be used meaningfully

[1] The usual notation convention when discussing complex numbers is to use letters in the first part of the alphabet $(a, b, c, d, \ldots)$ for real numbers and letters in the last part of the alphabet $(z, w, x, y, \ldots)$ for complex numbers. The latter usage can occasionally cause confusion with variables that range over the real numbers.

[2] Technically there are n solutions "with multiplicity." For example the equation $x^2 - 2x + 1 = 0$ has only $x = 1$ as a solution, but since $x^2 - 2x + 1 = (x - 1)(x - 1)$ this solution is counted twice.

and consistently to specify times relative to an arbitrary $t = 0$ moment. Similarly, even though negative length is meaningless, displacements and positions can be negative relative to an arbitrary 0 position.

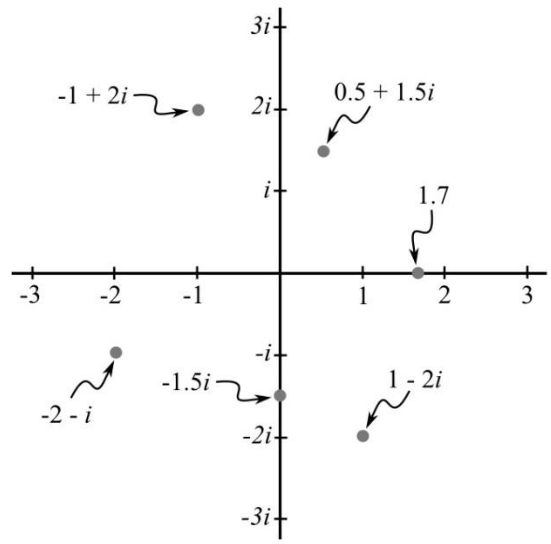

Fig. 2 Points corresponding to complex numbers on the complex plane

Because there was no obvious geometrical interpretation for complex solutions to equations, for many centuries complex numbers were thought to have no utility in modeling physical problems. Only at the beginning of the 19th century was it widely realized that, just as real numbers have a one-to-one correspondence with points on a line, complex numbers have a one-to-one correspondence with points on a plane. As shown in fig. 1, the complex plane has the standard real number line as its horizontal axis, while its vertical axis consists of an imaginary number line. Using these axes as a coordinate system, one can plot any complex number as a point on the plane. Real numbers lie along the horizontal axis, imaginary numbers along the vertical axis, and 0 is the origin.

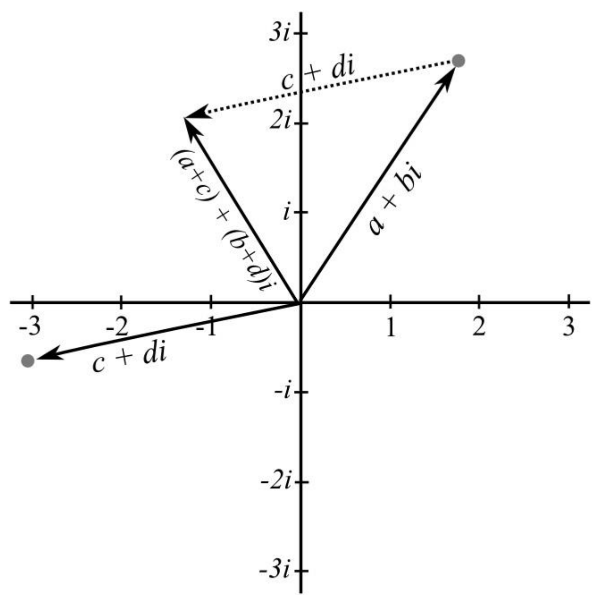

Fig. 1 Addition of complex numbers

The mapping of complex numbers to points on a plane leads naturally to the identification of a two-dimensional vector with each complex number, with the vector's magnitude and direction found by starting at the origin on the plane and drawing an arrow ending at the point associated with the number. As shown in Fig. 2, the vector, rather than the point, is now associated with the complex number, so that the complex number corresponds to a *displacement* on the complex plane rather than a specific location and need not be drawn starting at the origin. Since the real and imaginary parts of two complex numbers add separately, just like the components of a two-dimensional vector, the sum of the vectors corresponding to two complex numbers $z = (a + bi)$ and $w = (c + di)$, added in conventional fashion, results in the vector corresponding to the sum of these complex numbers: $z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$. This visualization helps to show that complex numbers, rather than containing "unreal" pieces, can be treated as an ordered pair of real numbers $a + bi \rightarrow (a, b)$, which add and subtract like vector coordinates.

Multiplying Complex-Plane Vectors

While the realization that complex numbers can stand in for two-dimensional vectors helps to make their deployment in physics more palatable, it doesn't immediately make them particularly useful; one might as well use other familiar two-dimensional vector notations for adding and subtracting them. The great utility of complex numbers arises from how they combine under multiplication. The rather messy rule for multiplying complex numbers algebraically, $zw = (a + bi)(c + di) = (ac - bd) + (bc + ad)i$, obscures an elegant geometrical fact that one can see only by considering the "length" of the associated vectors on the complex plane. Following the definition of conventional vectors, the length of a complex number $z = a + bi$, usually referred to as the **norm** or **modulus** of the number, is defined as $|z| = \sqrt{a^2 + b^2}$. The absolute value symbol is used in part as a reminder that the norm is always a positive real number and only 0 when $z = 0$. With this definition one sees that the norm of the product of complex numbers is the product of the norms:

$$|zw| = \sqrt{(ac - bd)^2 + (bc + ad)^2} = \sqrt{a^2c^2 - 2acbd + b^2d^2 + b^2c^2 + 2bcad + a^2d^2}$$

$$= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} = \sqrt{(a^2 + b^2)}\sqrt{(c^2 + d^2)} = |z||w|$$

Beginning physics students typically encounter two methods of multiplying vectors, dot products and cross products. The former can be used to combine two vectors of any matching dimension, but always produces a scalar, while the latter combines two vectors to make a vector, but can only be used on vectors that are three-dimensional or, somewhat surprisingly, seven-dimensional. The rule for complex number multiplication is a new way of combining two-dimensional vectors to get another two-dimensional vector.

In calculating the norm of a vector it is important to keep in mind that the square of the norm is not the square of the complex number: $(a + bi)^2 \neq a^2 + b^2$. But it is true that $(a + bi)(a - bi) = a^2 + b^2$. The complex number $(a - bi)$ is known as the **complex conjugate** of $(a + bi)$, and, conversely, $(a + bi)$ is the complex conjugate of $(a - bi)$. Symbolically, the complex conjugate of a number z is designated z^*, so that $zz^* = |z|^2$ and $(z^*)^* = z$. When looking at any expression for a complex number, one can always find its complex conjugate by changing all the i's to $-i$'s and $-i$'s to i's. Graphically, the complex conjugate of a vector on the complex plane is the mirror image of the vector reflected about the real axis.

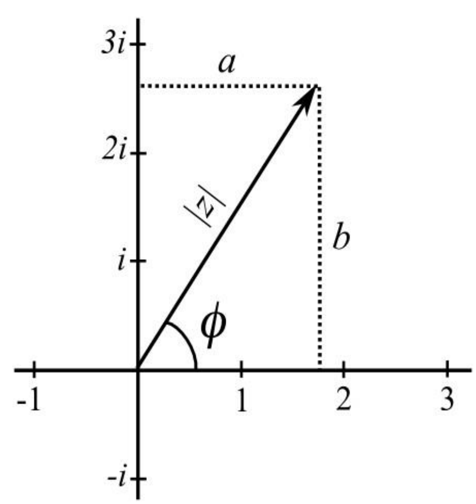

Fig. 3 Complex number in polar coordinates

While the magnitudes of two complex numbers have a simple relation to the magnitude of their product, it is less clear how the directions of the three corresponding vectors are related. Again there is a simple geometrical interpretation, but it is most apparent if the vectors are reexpressed in polar coordinates on the complex plane. As shown in fig. 3,

$$z = a + bi = (|z| \cos \phi) + (|z| \sin \phi)i,$$

where ϕ is the angle the vector associated with z makes with the positive real axis, often called the **argument** of z. This polar-coordinate expression is rather messy, but can be written in a much more elegant form using the **Euler relation**

$$e^{ix} = \cos x + i \sin x\,,$$

one of the most important formulas to keep in mind when dealing with complex numbers and provable by writing the Taylor expansions of each term.[3] Implementing this relation,

$$z = |z|(\cos\phi + i\sin\phi) = |z|e^{i\phi}$$

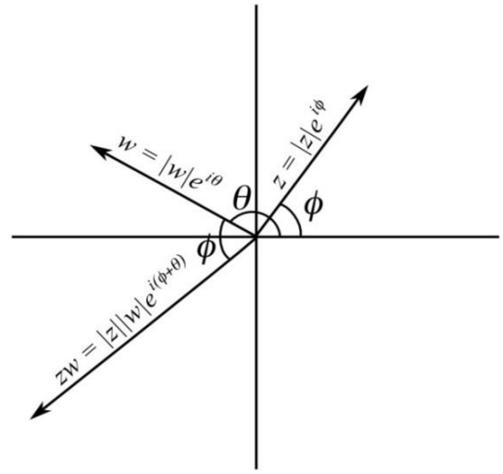

Fig. 4 Multiplication of complex numbers

In this form the visualization of complex-number multiplication on the complex plane is clearer, as shown in Fig. 4. The vector corresponding to the product $zw = |z|e^{i\phi}|w|e^{i\theta} = |z||w|e^{i(\phi+\theta)}$ is one with a norm equal to the *product* of z and w's norms and an argument that is the *sum* of z and w's arguments. In general, if one is adding complex numbers it is easier to do so if they are written in rectilinear $z = a + bi$ form, and if one is multiplying complex numbers it is easier to do so if they are written in polar $z = |z|e^{i\phi}$ form.

Application: AC Circuit Analysis Using Complex Phasors

We next examine a case where complex numbers – typically in their polar form – are frequently deployed in physics: AC circuits. Recall that an AC circuit is one where the applied voltage varies sinusoidally in time at some angular frequency ω: $V(t) = V_o\cos(\omega t)$. While one could equally well have used a sine function or other sinusoid, it is conventional to use a cosine function so that the applied voltage is at its maximum **amplitude** V_o at time $t = 0$.

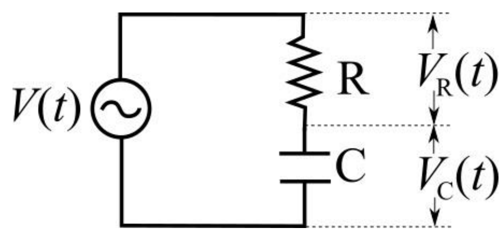

Figure 5 RC circuit with applied AC voltage

Consider this voltage applied to a resistor R and capacitor C wired in series. From Kirchoff's laws we know that at any moment in time the voltage drop across the resistor $V_R(t)$ and the voltage drop across the capacitor $V_C(t)$ must sum to equal the applied voltage:

$$V(t) = V_R(t) + V_C(t)$$

[3] Taking the complex conjugate of both sides of the Euler relation yields $e^{-ix} = \cos x - i\sin x$, which can be added or subtracted from the original relation to yield widely used complex expressions for sine and cosine: $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$.

From physical considerations, we know that if the applied voltage is periodic with angular frequency ω then these voltage drops must also be periodic with the same angular frequency; a general expression for each of these drops is then one that includes an unknown amplitude and phase:

$$V_R(t) = V_{R_o}\cos(\omega t + \varphi_R) \quad \text{and} \quad V_C(t) = V_{C_o}\cos(\omega t + \varphi_C).$$

Similarly the current through the circuit will also be periodic with the same frequency and can be written $I(t) = I_o\cos(\omega t + \varphi_I)$. From further physical considerations, it can be shown that the amplitudes and phases are interrelated in the following way:

$$V_{R_o} = \mathrm{R}I_o \qquad V_{C_o} = \frac{I_o}{\omega\mathrm{C}} \qquad \varphi_R = \varphi_I \qquad \varphi_C = \varphi_I - \tfrac{\pi}{2}.$$

The Kirchoff voltage-drop equation therefore may be rewritten as

$$V_o\cos(\omega t) = \mathrm{R}I_o\cos(\omega t + \varphi_I) + \tfrac{I_o}{\omega\mathrm{C}}\cos\left(\omega t + \varphi_I - \tfrac{\pi}{2}\right)$$

The mathematical task is to solve this equation for I_o and φ_I in terms of V_o, R, C, and ω. This solution can then be used to find the amplitudes and phases for the voltages across the resistor and capacitor. While the solution to this equation can be found using trigonometric identities, a faster and more intuitive path is to map this problem onto the complex plane. We begin by noting that, from Euler's relation, each cosine function constitutes the real part of a complex exponential function, e.g.,

$$I_o\cos(\omega t + \varphi_I) = \mathrm{Re}\left[I_o e^{i(\omega t + \varphi_I)}\right].$$

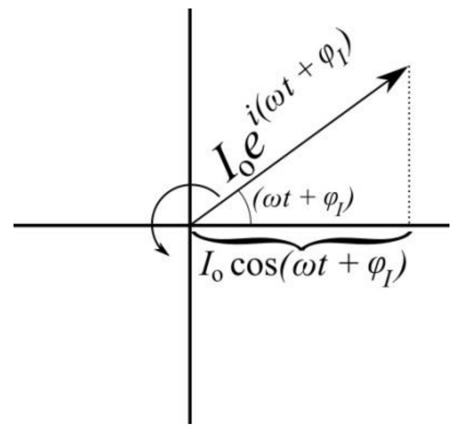

Figure 6 Phasor rotating on the complex plane

Graphically, the complex exponential is represented by a vector in the complex plane, known in this context as a **phasor**, which has a constant magnitude equal to the amplitude of the cosine and rotates counterclockwise in time at angular frequency ω, thus making one full revolution every $2\pi/\omega$ seconds. The real part of this vector at any moment in time is its component along the real axis, which oscillates from a maximum of I_o to a minimum of $-I_o$ and is zero at moments when the vector points along the imaginary axis. The continually changing angle this vector makes with the positive real axis is equal to the continually changing argument of the cosine function.

While this rotating phasor alone can provide a useful visualization of the cosine function, its real utility is in analyzing relationships between cosine functions. Returning to the voltage-drop equation, consider the corresponding equation between complex exponentials:

$$V_o e^{i\omega t} = \mathrm{R}I_o e^{i(\omega t + \varphi_I)} + \frac{I_o}{\omega\mathrm{C}} e^{i\left(\omega t + \varphi_I - \frac{\pi}{2}\right)}.$$

In order for this equality to hold, both the real parts of each side and the imaginary parts of each side must be equal. The real-part equality is equivalent to the voltage-drop equation, so solutions of this complex equation for I_o and φ_I will also be solutions for the voltage-drop equation.

On first glance adding in an extra imaginary part to the equality would seem to complicate matters even further; however, this approach has several advantages. First, the arguments of the exponentials are much easier to algebraically manipulate than the arguments of cosine functions. Here one can eliminate the time-dependent part of the equation by dividing by $e^{i\omega t}$, whereas a similar elimination in the cosine equation requires cumbersome trig-identity expansions. Second, the graphical representations of the complex phasors often lend themselves to simple geometric solutions to algebraic problems. In this case we first draw the three phasors representing each term: the applied voltage, the voltage across the resistor, and the voltage across the capacitor. As before, each phasor rotates counterclockwise with the same angular speed ω, so the relative angles between them remain the same, with the angle between the two phasors corresponding to voltage drops across the resistor and capacitor always being 90°. From this picture, one can see that the complex exponential equation corresponds to a vectorial sum of two legs of a right triangle equaling its hypotenuse. One can then use familiar geometric relations to extract the quantities in question from the magnitudes of each vector:

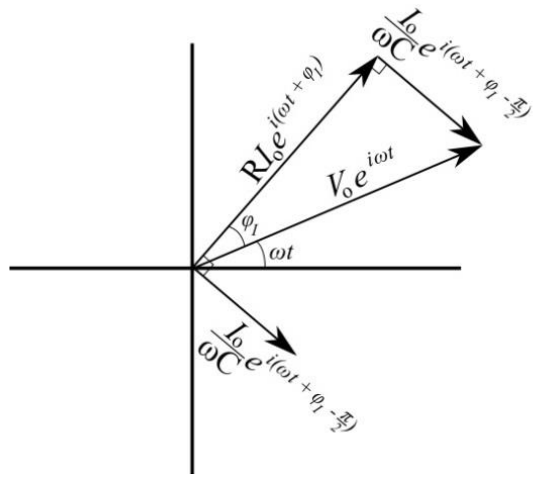

Figure 7 Addition of phasors for the RC circuit.

$$(V_o)^2 = (\mathrm{R}I_o)^2 + \left(\frac{I_o}{\omega \mathrm{C}}\right)^2$$

$$\Rightarrow I_o = \frac{V_o}{\sqrt{\mathrm{R}^2 + \left(\frac{1}{\omega \mathrm{C}}\right)^2}} \qquad \varphi_I = \tan^{-1}\left(\frac{I_o/\omega \mathrm{C}}{\mathrm{R}I_o}\right) = \tan^{-1}\left(\frac{1}{\omega \mathrm{RC}}\right)$$

Any equation involving sums and differences of sinusoids can be tackled in a similar fashion. First, write all sinusoids as cosines (i.e. using $\cos(x - \pi/2) = \sin x$), then invoke the complex mapping $A\cos(x + \varphi) \to Ae^{i(x+\varphi)}$, and finally solve the complex equation algebraically or geometrically keeping in mind that the real parts and imaginary parts equate.

Application: AC Circuit Power & Multiplying Complex Phasors

In some texts, the distinction between the real voltage function and its complex mapping is notationally distinguished. For example, the actual applied voltage is specified by $V = V_o \cos(\omega t)$ and the complex-mapping counterpart by $\tilde{V} = V_o e^{i\omega t}$, a notational distinction we follow here. However, the mapping of cosines to complex exponentials is so ubiquitous that authors may lapse into statements such as, "the applied voltage is given by $V_o e^{i\omega t}$," although properly speaking the voltage is given by the real part of this expression. Nevertheless, the reader should keep in mind at all times that this ubiquity is not a mathematical blank check to replace cosines by complex exponentials in all situations. While the complex mapping works in a straightforward manner when solving sums and differences of cosines, the additional imaginary part must be handled carefully when dealing with products and quotients of cosines.

For example, consider the power dissipated by the circuit in question, given by the product of the applied voltage and current: $P(t) = I(t)V(t) = I_o \cos(\omega t + \varphi_I) V_o \cos(\omega t)$. This expression can again be simplified using trig identities, but we can also approach it with the complex mapping using the fact that, for any complex number z, Re $[z] = \frac{1}{2}(z + z^*)$. With $\tilde{V} = V_o e^{i\omega t}$ and $\tilde{I} = I_o e^{i(\omega t + \varphi_I)}$, we thus have

$$P(t) = I(t)V(t) = \frac{1}{2}(\tilde{I} + \tilde{I}^*)\frac{1}{2}(\tilde{V} + \tilde{V}^*) = \frac{1}{2}\left(I_o e^{i(\omega t+\varphi_I)} + I_o e^{-i(\omega t+\varphi_I)}\right)\frac{1}{2}\left(V_o e^{i\omega t} + V_o e^{-i\omega t}\right)$$

$$= \frac{1}{4} I_o V_o \left(e^{i(2\omega t+\varphi_I)} + e^{-i(2\omega t+\varphi_I)} + e^{i\varphi_I} + e^{-i\varphi_I}\right) = \frac{1}{2} I_o V_o (\cos(2\omega t + \varphi_I) + \cos\varphi_I)$$

For this problem, the transition to complex numbers offers little computational efficiency over using trig identities on the original product of cosine functions. However, there are many problems where the efficiency gains of making this transition are considerable. The advantage of the complex formulation when dealing with products is principally that it decomposes expressions into sums of complex exponentials with well-defined frequencies, which not only can simplify further calculations but can give one an intuitive sense of the oscillation rates of different parts of a result.

As a simple example, we note that one is rarely interested in the instantaneous power $P(t)$ dissipated by a circuit. Rather one is concerned with the average power dissipated over time. Formally, the average power is defined as $\bar{P} \equiv \frac{1}{T}\int_0^T P(t)\,dt$, where T is the time scale one wishes to averages over. For AC electrical circuits driven by a single source it is sufficient to average over a single period of oscillation $T = \frac{2\pi}{\omega}$ since all dynamic properties repeat on this time scale, but in other situations one may wish to average over multiple periods, over time-scales set by a measurement apparatus, or in the limit $T \to \infty$. From the expression above, $P(t) = \frac{1}{4} I_o V_o \left(e^{i(2\omega t+\varphi_I)} + e^{-i(2\omega t+\varphi_I)} + e^{i\varphi_I} + e^{-i\varphi_I}\right)$, one can immediately identify the first two terms as oscillations with periods $\frac{2\pi}{2\omega}$ which will average to zero over the period of the applied voltage, leading to an average power of $\bar{P} = \frac{1}{4} I_o V_o \left(e^{i\varphi_I} + e^{-i\varphi_I}\right) = \frac{1}{2} I_o V_o \cos\varphi_I$.

Application: Harmonic Wave Superpositions

Along with AC circuits, another common implementation of the mapping of cosines to complex exponentials in physical problems is found in the analysis of waves. As discussed further in a later section, functions describing waves can often be decomposed into sums or integrals of harmonic (i.e. sinusoidal) components. In the case of a one-dimensional wave, each harmonic wave has the form $A\cos(kx - \omega t + \varphi)$, where A is the **amplitude** of the harmonic wave, k the **wavenumber** ($k \equiv \frac{2\pi}{\lambda}$, with λ the wavelength), ω the angular frequency, and φ a constant phase necessary to account for the choice of origins of position and time.

As a simple example, consider a wave made up of two harmonic waves (often described as the **superposition** of two harmonic waves):

$$g(x,t) = A_1\cos(k_1x - \omega_1t + \varphi_1) + A_2\cos(k_2x - \omega_2t + \varphi_2)$$

In the laboratory such a wave might be produced by starting with two harmonic waves that are directed to physically overlap, resulting in this **interference** of the harmonic waves. In this case our mapping of cosines to complex exponentials results in the function

$$\tilde{g}(x,t) = A_1e^{i(k_1x-\omega_1t+\varphi_1)} + A_2e^{i(k_2x-\omega_2t+\varphi_2)}$$

whose real part corresponds to the real wave.

Why would one go to the trouble of using the complex mapping? A significant advantage in the case of waves stems from the fact that particular harmonic components of a wave may be subjected to physical perturbations which alter their phase. For example, suppose the first harmonic wave is re-directed to reflect off a barrier before superposing with the second one. The additional distance L it travels will add a phase shift of k_1L, while the reflection will typically flip the sign of the amplitude, or, equivalently, add a phase of π: $A_1\cos(k_1x - \omega_1t + \varphi_1)$ is thus altered to $A_1\cos(k_1(x+L) - \omega_1t + \varphi_1 + \pi) = A_1\cos(k_1x - \omega_1t + \varphi_1 + \pi + k_1L)$. While this alteration is straightforward to write down by inspection, it is an algebraically complicated operation to change the argument of the cosine. In the complex mapping, however, this alteration is accomplished through simple multiplication: $A_1e^{i(k_1x-\omega_1t+\varphi_1)} * e^{i(k_1L+\pi)} = A_1e^{i(k_1x-\omega_1t+\varphi_1+\pi+k_1L)}$. For many analyses of wave phenomena the short term cost of going to the trouble of mapping cosines to complex exponentials is a worthwhile investment in the long run due to the algorithmic ease with which one can account for various phase shifts.

We reiterate the warning, a version of which was first mentioned in connection with the complex exponential mapping of AC circuits, that many authors will make no notational distinction between the purely real function describing a harmonic wave $g(x,t)$ and its complex counterpart $\tilde{g}(x,t)$. In these texts the complex functions are the only ones discussed and it is assumed the reader can translate these functions back to their real counterparts as needed. Another notational technique (occasionally employed for AC circuits, but more widely used in wave analysis) is to include any constant complex phase shifts in the formerly real amplitude of the wave. In this case, $A_1e^{i(k_1x-\omega_1t+\varphi_1+\pi+k_1L)} = \tilde{A}_1e^{i(k_1x-\omega_1t)}$, where $\tilde{A}_1 = A_1e^{i(\varphi_1+\pi+k_1L)}$ is the **complex amplitude** of the harmonic wave. In addition to making the expression for the wave more compact, this choice allows one to dedicate the explicit space and time dependence of the wave to the complex exponential while encoding all information about both the wave's amplitude and relative phase in the norm and argument of the single complex constant $\tilde{A}_1$.

Application: Intensity of Harmonic Wave Superpositions

Just as with AC circuits, one must use a bit of extra caution when using complex expressions for waves in calculations requiring products. For example, the time-dependent **intensity** of a wave is proportional to the square of its magnitude: $I(t) = c[g(x,t)]^2$, where c is a constant that depends on the types of waves being analyzed. To calculate the intensity using the complex function $\tilde{g}(x,t)$ and its complex conjugate $\tilde{g}^*(x,t)$ we can use the same relation employed in the calculation of power for AC circuits, namely $g(x,t) = \frac{1}{2}\left(\tilde{g}(x,t) + \tilde{g}^*(x,t)\right)$, so that $I(t) = \frac{c}{4}\left(2|\tilde{g}(x,t)|^2 + [\tilde{g}(x,t)]^2 + [\tilde{g}^*(x,t)]^2\right)$. For the complex version of the superposition of two harmonic waves: $\tilde{g}(x,t) = \tilde{A}_1 e^{i(k_1 x - \omega_1 t)} + \tilde{A}_2 e^{i(k_2 x - \omega_2 t)}$ with complex amplitudes $\tilde{A}_1 = A_1 e^{i\varphi_1}$ and $\tilde{A}_2 = A_2 e^{i\varphi_2}$ the time-dependent intensity is then

$$I(t) = \frac{c}{4}\Big(2\left[\left|\tilde{A}_1\right|^2 + \left|\tilde{A}_2\right|^2 + \tilde{A}_1 \tilde{A}_2{}^* e^{i\left((k_1-k_2)x-(\omega_1-\omega_2)t\right)} + \tilde{A}_1{}^* \tilde{A}_2 e^{-i\left((k_1-k_2)x-(\omega_1-\omega_2)t\right)}\right] \\ + \left[\tilde{A}_1{}^2 e^{i(2k_1x-2\omega_1 t)} + \tilde{A}_2{}^2 e^{i(2k_2x-2\omega_2 t)} + 2\tilde{A}_1\tilde{A}_2 e^{i\left((k_1+k_2)x-(\omega_1+\omega_2)t\right)}\right] \\ + \left[\tilde{A}_1{}^{*2} e^{-i(2k_1x-2\omega_1 t)} + \tilde{A}_2{}^{*2} e^{-i(2k_2x-2\omega_2 t)} + 2\tilde{A}_1{}^*\tilde{A}_2{}^* e^{-i\left((k_1+k_2)x-(\omega_1+\omega_2)t\right)}\right]\Big).$$

While this expression looks extremely cumbersome, it can be simplified somewhat by rewriting sums of complex exponentials into cosines. More importantly, as with the time-dependent power of AC circuits, this instantaneous intensity is rarely a relevant quantity since any instrument records an average intensity[4] over a finite time scale $\bar{I} \equiv \frac{1}{T}\int_0^T I(t)\,dt$, and this averaging will lead to the elimination of many terms.

As a familiar example, one's eyes and ears can sense intensity changes in light and sound on the order of a tenth of a second. Most audible harmonic sound waves ("pure tones") oscillate hundreds or thousands of times per second, while visible harmonic light waves ("monochromatic light") oscillate on the order of 10^{14} times per second. Any sinusoid or complex exponential oscillating in $I(t)$ at these frequencies will thus average to zero on our perceptive time scales. In the above expression, any term oscillating at multiples or sums of ω_1 and ω_2 vanishes on time averaging, meaning the 2nd and 3rd bracketed terms disappear. Similarly, under conditions where any individual harmonic frequency in a superposed wave oscillates faster than the time interval over which the intensity is averaged, all terms in $[\tilde{g}(x,t)]^2$ and $[\tilde{g}^*(x,t)]^2$ average to zero, leaving $\bar{I} = \frac{c}{2}\overline{|\tilde{g}(x,t)|^2}$.

Examining this remaining term to be time averaged for the problem at hand, we have

$$\bar{I} = \frac{c}{2}\left[\overline{\left|\tilde{A}_1\right|^2 + \left|\tilde{A}_2\right|^2 + \tilde{A}_1\tilde{A}_2{}^* e^{i\left((k_1-k_2)x-(\omega_1-\omega_2)t\right)} + \tilde{A}_1{}^*\tilde{A}_2 e^{-i\left((k_1-k_2)x-(\omega_1-\omega_2)t\right)}}\right] \\ = \frac{c}{2}\left[\overline{A_1{}^2 + A_2{}^2 + A_1 e^{i\varphi_1} A_2 e^{-i\varphi_2} e^{i\left((k_1-k_2)x-(\omega_1-\omega_2)t\right)} + A_1 e^{-i\varphi_1} A_2 e^{i\varphi_2} e^{-i\left((k_1-k_2)x-(\omega_1-\omega_2)t\right)}}\right] \\ = \frac{c}{2}\left[\overline{A_1{}^2 + A_2{}^2 + 2A_1A_2\cos\left((k_1-k_2)x-(\omega_1-\omega_2)t+\varphi_1-\varphi_2\right)}\right]$$

There are three possibilities for the time-averaging of these remaining terms. In the first case not only are the frequencies ω_1 and ω_2 fast compared with the time averaging, but the frequencies are far enough apart that their difference is as well. In this case the oscillating terms vanish and the time-averaged intensity is $\frac{c}{2}\left(A_1{}^2 + A_2{}^2\right)$, meaning the intensity of the combined waves is equal to the sum of the intensities of each individual harmonic wave. In the second case, the difference in frequencies is small enough that the oscillating intensity can be detected. The intensity will wax and wane at a frequency $(\omega_1 - \omega_2)$, known as the **beating frequency**. Such an oscillation in intensity can be observed directly with the ear when two pure tones only a few Hz apart are played simultaneously. In the third case, the frequencies are identical, in which case the wavenumbers are also almost invariably identical. The time-averaged intensity in this case is $\bar{I} = \frac{c}{2}\left[A_1{}^2 + A_2{}^2 + 2A_1A_2\cos(\varphi_1-\varphi_2)\right]$, so that the difference in

[4] Unfortunately the word "intensity" and the symbol I are used both in reference to the instantaneous $I(t)$ and the time-averaged intensity, with the latter quantity more often the one that is meant.

phases determines whether the **interference** of two harmonic waves result in an intensity that is larger or smaller than the sum of the intensities of the individual waves.

Using trigonometric identities, these and similar results can all be arrived at using only the real functions describing the harmonic waves. However, the relatively compact $\bar{I} = \frac{c}{2}\overline{|\tilde{g}(x,t)|^2}$ formula that applies under the specified conditions typically makes the translation to the complex formalism worthwhile.

Application: Fourier Series and Transforms

Harmonic waves and their superpositions are especially useful topics of study since one can show that any periodic wave can be written as a – perhaps infinite – superposition of harmonic waves. More generally, any real analytic function $f(x)$ that is periodic, i.e. $f(x) = f(x + \lambda)$ for some constant λ, can be written as a sum of signs and cosines:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(2\pi n \frac{x}{\lambda}\right) + b_n \sin\left(2\pi n \frac{x}{\lambda}\right) = \sum_{n=0}^{\infty} a_n \cos(k_n x) + b_n \sin(k_n x)$$

where $k_n \equiv \frac{2\pi n}{\lambda}$ and the real a_n and b_n coefficients depend on the particular periodic wave in question. This **Fourier series** decomposition of a periodic wave can alternately be written in a more compact form, using complex exponentials and Euler's relation, as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{ik_n x}\,,$$

where the (possibly complex) c_n coefficients again depend on the particular periodic wave in question.[5] We emphasize that in this equation the complex exponential sum is an exact equality of the sum of sinusoids and not a mapping from cosines to complex exponentials as it was in the previous cases. The use of complex exponentials in this case has some advantages in compactness and calculation and so is a widely used "decomposition" of periodic functions describing physical phenomena.

As the interval λ is increased, the spacing between successive k_n's gets smaller and smaller, and in the limit $\lambda \to \infty$ the discrete k_n's become a continuous variable k. Similarly, the discrete coefficients c_n in one-to-one correspondence with each k_n also become continuous and are now written as a function of k, $c(k)$. Finally, with both k and $c(k)$ now continuous, the formerly discrete sum decomposition is now an integral:

$$f(x) = \int_{-\infty}^{\infty} c(k)\, e^{ikx} dk$$

In this limit, the function $c(k)$, which specifies the amplitude of each sinusoidal component that $f(x)$ is decomposed into, is known as the **Fourier transform** of $f(x)$. The advantage of taking this limit is that

[5] The relations between the a_n's, b_n's, and c_n's that must hold for this equality, as well as the relations between c_n's that ensure the sum is purely real are left as exercises for the reader.

now one is not limited to decomposing periodic functions into sinusoidal components, but can describe a large class of non-periodic functions just in terms of continuous sums of complex exponentials. The Fourier transform, like the Fourier series, can be recast entirely in terms of real sines and cosines, but in practice this is rarely done, in part because, even in the case of using real functions and coefficients, the process of determining these coefficients often requires the mathematical machinery of doing integrals on the complex plane, a process that is a central concern of the field of **complex analysis**. The complex coefficients $c(k)$ also typically require such integrals; in fact it can be shown that if the function $f(x)$ is known, the coefficients are given by the compact but often tricky-to-evaluate equation

$$c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$$

Application: Quantum Mechanical Waves

The use of complex numbers is at its most ubiquitous in physics in the field of quantum mechanics. Indeed, even the most basic quantum equations, such as the Schrödinger equation, are written in terms of complex numbers. As with the applications discussed above, the use of complex numbers is not a necessity for doing quantum calculations, but reformulating quantum algorithms in terms of only real numbers and functions would be so mathematically cumbersome that in practice a complex formalism is always used. An accounting of basic methods of how complex numbers are employed to encode and predict experimental information is the central topic of an introductory quantum mechanics course, and thus beyond the scope of these notes. However, we do wish to touch on the topic of simple quantum waves, since they share a considerable formal overlap with the complex exponential mapping of classical waves, an overlap which can often lead to considerable conceptual confusion for students both during and after their introduction to the formalism of quantum mechanics.

Quantum mechanics posits that, in the ideal, isolated case, every object is associated with a typically complex-valued **wavefunction** over space and time $\Psi(x,t)$.[6] The physical meaning of this wavefunction itself is a hotly debated topic, but at minimum it is generally agreed that it serves as a way of encoding information about possible outcomes of measurements made on the object it is associated with. As an example, if one measures the position of an object, the outcome of this measurement is not deterministic; rather, the probability of finding the object at time t somewhere between $x = a$ and $x = b$ is given by the integral $\int_a^b P(x,t)dx$, where $P(x,t)$ is a function known as the **probability density** of the object's location. The relation between this probability density and the object's wavefunction is simply $P(x,t) = |\Psi(x,t)|^2$.

One of the simplest class of wavefunctions is that of a **free particle**, so called because the object is subjected to no outside forces. The simplest free particle wavefunction has the form

$$\Psi(x,t) = Ae^{i(kx-\omega t)}\,,$$

[6] As with our discussion of functions describing classical waves, we focus on one-dimensional waves for simplicity.

where k and ω are real constants, but A can be complex. Note that this wavefunction is formally identical to the complex *mapping* of a cosine describing a classical harmonic wave $h(x,t) = A\cos(kx - \omega t + \varphi)$ $\rightarrow \tilde{h}(x,t) = \tilde{A}e^{i(kx-\omega t)}$. While there are important historical and physical reasons why these two expressions are identical, it is imperative for one to be aware of the differences in what they mean and how they are used in order to avoid erroneously applying intuitions developed about the classical wave to the quantum one.

In the classical case, the real amplitude A corresponds to a physical quantity such as air pressure (sound waves), water height (water waves), or electric field (light waves). The wave itself is an oscillation of this quantity in space and time, with the positive constants k and ω specifying the spatial and temporal periodicities of these oscillations respectively. The minus sign in the $kx - \omega t$ expression results in the wave moving in the positive x direction (a wave moving in the negative x direction requires a plus sign). The time-averaged intensity, or energy density, carried by the wave is typically proportional to the square of the norm of the complex mapping, which for a harmonic wave is just the square of the amplitude, $\left|\tilde{h}(x,t)\right|^2 = \left|\tilde{A}\right|^2$.

For the quantum wave, as intimated above, the possibly complex amplitude A itself has no apparent physical interpretation. Nor is the complex exponential standing in for a real cosine function that corresponds to something physical – rather the imaginary piece is equally indispensable when using the wavefunction in established quantum algorithms. In particular the probability density for this free particle is $P(x,t) = |\Psi(x,t)|^2 = |A|^2$. This result is formally similar to the classical wave's time-averaged intensity, but there is no explicit time-averaging taking place. More surprisingly, this probability density is a constant independent of space and time, so a free particle described by a complex exponential is at any moment equally likely to be found anywhere. Consequently there is nothing dynamic, wave or particle, that is 'moving to the right' for an object described by this wavefunction; rather, it describes what is known as a **stationary state**.[7] Despite this fact, the formal analogy with the classical wave is so strong that many authors misleadingly refer to objects described by this complex exponential as "moving to the right." The reader is strongly advised to resist the physical picture this phrase implies when developing their intuitions about quantum behavior.

[7] To quantum mechanically describe a free particle moving to the right requires a superposition of these complex exponential wavefunctions.