Today

I Last time

II Maxwell's Equations

III Electromagnetic Waves

Optics

Day 8

Feb 23rd, 2017

P1/5

I • Introd the wave eqn:

$$\frac{\partial^2\psi}{\partial t^2} - v^2 \frac{\partial^2 \psi}{\partial x^2} = 0, \quad \psi = \psi(x,t).$$

and its harmonic solutions

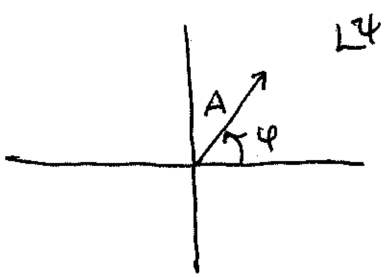
$$\psi = A e^{i(\omega t - kx + \epsilon)} = A e^{i\varphi}$$

which are characterized by

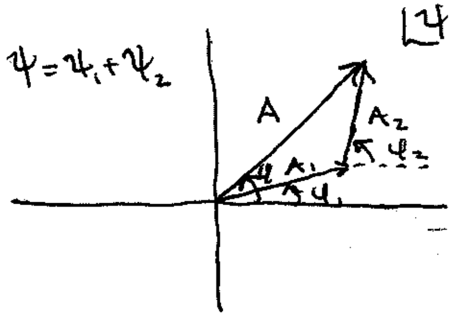
A, $\omega = \frac{2\pi}{\tau} = 2\pi\nu = 2\pi f$,

$k = \frac{2\pi}{\lambda}$ and ϵ "initial phase"

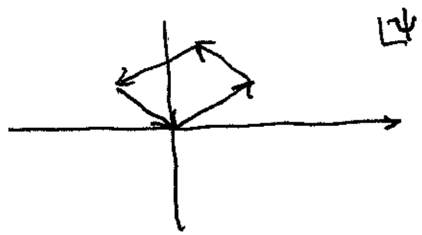
• Briefly studied the phasor representation $\psi = Ae^{i\varphi}$



$\psi = \psi_1 + \psi_2$



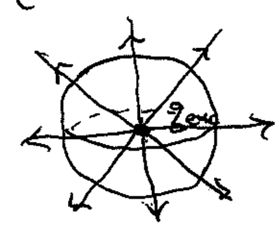
Characterized deconstructive interference



II We will briefly review Maxwell's equations in two forms: the integral form and the differential one.

Gauss' law: One of the first results you encountered in E&M

$$\underbrace{\oint_A \vec{E}\cdot d\vec{S}} = \frac{q_{enc}}{\epsilon_0}$$



Electric flux measures how much electric field pierces a surface A.

We'd like to recast this is a local form that only refers to one point.

To do so, introduce <u>vector derivatives</u>, simple to define: Suppose $\vec{A} = \vec{A}(t)$ then

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{x} + \frac{dA_y}{dt}\hat{y} + \frac{dA_z}{dt}\hat{z}.$$

Not hard to prove that

$$\frac{d}{dt}(a\vec{A}) = \frac{da}{dt}\vec{A} + a\frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

and

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

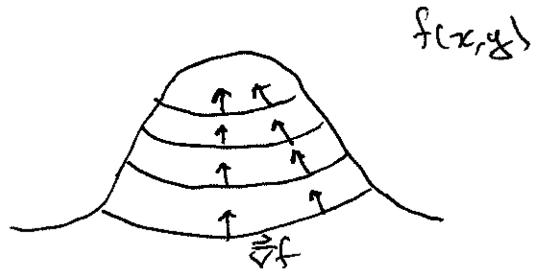
Life gets more interesting when $\vec{A} = \vec{A}(x,y,z)$, then we have $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, which we collect into a vector of their own:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

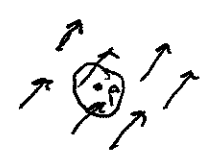
A vector of derivatives allows us to consider a wealth of operations. Given $f(x,y)$ we can take

$$\vec{\nabla}f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

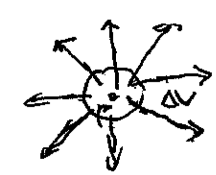
This is the <u>gradient</u> of f and points in the direction of steepest increase of f.



If we're given a vector field $\vec{A}(x,y,z)$, e.g.



or



We can form

$$\vec{\nabla}\cdot\vec{A}\Big|_P = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

This measures how much of the vector field leaves a small region ΔV around P, how $\vec{A}$ "diverges" per unit volume. The divergence allows us to recast Gauss' law at a point. Consider

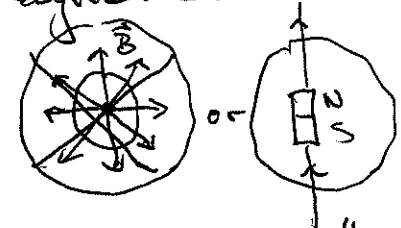
$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\!\!\oint_A \vec{E} \cdot d\vec{s} = \lim_{\Delta V \to 0} \frac{q_{enc}}{\Delta V \epsilon_0}$$

The left hand side is $\vec{\nabla} \cdot \vec{E}$ and so

$$\boxed{\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0} \quad \text{Gauss' law}$$

The second Maxwell equation can be analyzed in a very similar manner; this is Gauss' law for magnetism

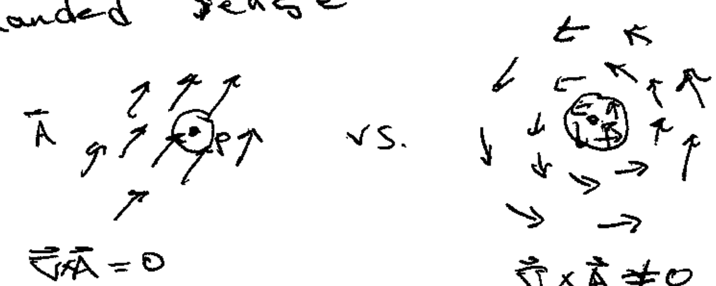
$$\oint\!\!\oint_A \vec{B} \cdot d\vec{s} = 0$$



Read as "No magnetic monopoles" or as magnetic fields are created by two pole magnets. Then

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\!\!\oint_A \vec{B} \cdot d\vec{s} = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

This derivative measures how much a vector field circulates around a point in a right handed sense

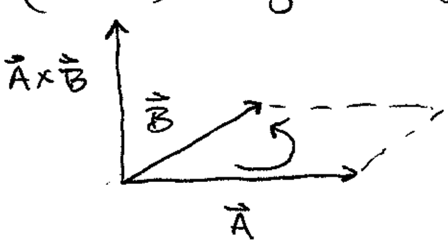


This operation allows us to caste the last two Maxwell equations in

You'll recall $\vec{A} \times \vec{B}$ and

$|\vec{A} \times \vec{B}|$ = are of parallelogram spanned by $\vec{A}$ and $\vec{B}$

dir$(\vec{A} \times \vec{B})$ = given by right hand rule

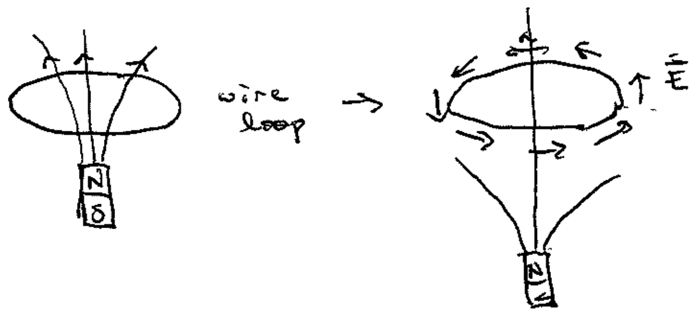


We can also introduce, for $\vec{A} = \vec{A}(x,y,z)$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

differential form. For Faraday's law

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_A \vec{B}\cdot d\vec{S} = -\iint_A \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$



So,

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A}\oint_c \vec{E}\cdot d\vec{l} = \lim_{\Delta A \to 0} \frac{-1}{\Delta A}\iint_A \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

gives

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

By similar reasoning, the Ampère-Maxwell eqn

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0\epsilon_0 \frac{d}{dt}\iint_A \vec{E}\cdot d\vec{s} + \mu_0 \iint_A \vec{J}\cdot d\vec{s}$$

gives

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}}$$

or in vacuum

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Remarkably, combining these four equations we can show that there are electromagnetic waves!

III Recall the BAC-CAB rule

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}).$$

A similar result holds for $\vec{\nabla}$,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \underbrace{\nabla^2}_{\text{short for } \vec{\nabla}\cdot\vec{\nabla}}\vec{A}.$$

Let's apply this to our last result

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu_0\epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

or

$$\vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{B}}}_{0 \text{ Gauss' law}}) - \nabla^2\vec{B} = -\mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

So that

$$\nabla^2\vec{B} = \mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

that is

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0\epsilon_0}\left(\frac{\partial^2 \vec{B}}{\partial x^2} + \frac{\partial^2 \vec{B}}{\partial y^2} + \frac{\partial^2 \vec{B}}{\partial z^2}\right)$$

This is the wave equation (!) with

$v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$ and where each component of $\vec{B}$, e.g. B_x, is like our previous ψ. Not only that, but the waves are in 3D too.