

Homework 7

Due Friday, November 13th in class

If you haven't already, read Chapter 2 of Schroeder's book *Thermal Physics*.

The goal of this problem set is to connect the two formulations of entropy that we have been discussing in class. When we first started studying entropy we introduced the thermodynamic notion of the "quality of heat": $\Delta S = \int_a^b \frac{dQ}{T}$. Instead the statistical notion of entropy is: $S = k \ln \Omega$, where Ω is the multiplicity of the macrostate under consideration. How are these related?

In this problem set you will show the relation explicitly for an idea gas.

Problem 1 (Another form of Stirling's approximation)

First we need a mathematical tool. In class we discussed Stirling's approximation:

$$n! \approx \sqrt{2\pi n} e^{-n} n^n.$$

(a) Take the natural logarithm of this expression and prove that:

$$\ln n! \approx n \ln n - n + \frac{1}{2} \ln(2\pi n).$$

(b) Use Stirling's approximation to calculate: $\ln(100!)$, $\ln(10^6!)$ and $\ln(10^{23}!)$. Explain why we often drop the last term of your answer from part (a) and only use

$$\ln n! \approx n \ln n - n.$$

Problem 2 (In your own words)

Read section 2.5 of Schroeder for a second time. Close your book and explain why the multiplicity Ω_1 is proportional to the volume of momentum space V_p in your own words.

Problem 3 (Sackur-Tetrode Equation)

Using your result from Problem 1 and Schroeder's result in equation (2.40), derive the Sackur-Tetrode equation

$$S = Nk \left[\ln \left(\frac{V}{N} \left(\frac{4\pi mU}{3Nh^2} \right)^{3/2} \right) + \frac{5}{2} \right],$$

from the statistical definition of entropy $S = k \ln \Omega$. Here h is Planck's constant.

Problem 4 (Deriving the Entropy-Heat Relation)

Using the Sackur-Tetrode equation, show that during the reversible isothermal expansion of a monatomic ideal gas, the change in entropy is related to the heat input Q by the simple formula

$$\Delta S = \frac{Q}{T}.$$

You have just derived our thermodynamic notion of entropy from our statistical notion in the special case of an ideal gas and an isothermal process.

We won't prove it in this course, but this formula is valid for any reversible process. Show, however, that it is not valid for the free expansion process that we discussed in class.

Problem 5 (Validity)

As you know, you made several approximations in your derivation of the Sackur-Tetrode equation. In particular, according to the Sackur-Tetrode equation, the entropy of a monatomic ideal gas can become negative when its temperature (and hence its energy) is sufficiently low. Of course this is absurd. (a) Explain why. So, the Sackur-Tetrode equation must be invalid at very low temperatures. Suppose you start with a sample of helium at room temperature and atmospheric pressure, then lower the temperature holding the density fixed. Pretend that the helium remains a gas and does not liquefy. (b) Below what temperature would the Sackur-Tetrode equation predict that S is negative?

The behavior of gases at very low temperatures is fascinating and led to a nobel prize for the creation of a Bose-Einstein condensate in recent years. This is one of the topics that is often studied in the Thermal Physics course.

Problem 6 (An example)

Use the Sackur-Tetrode equation to calculate the entropy of a mole of argon gas at room temperature and atmospheric pressure. Why is the entropy greater than that of a mole of helium under the same conditions?