Modern
Day 13

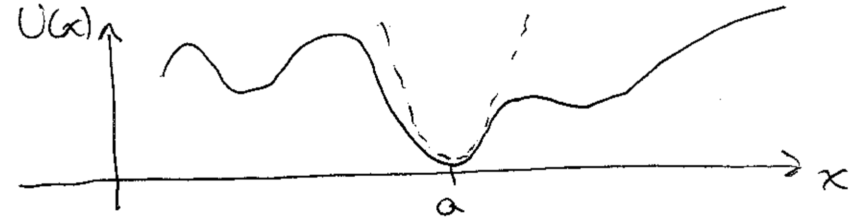
Outline

I Last time

II Wave motion of materials

III Superposition and group velocity

I We argued that near a stable equilibrium of a potential

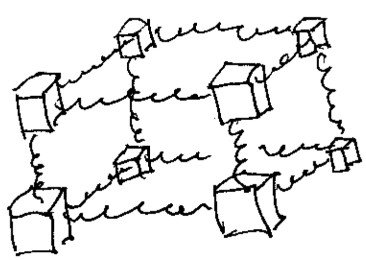


$$U(x) \simeq U(a) + 0 + \tfrac{1}{2} U''(a)\,(x-a)^2$$

it always looks like a harmonic oscillator with

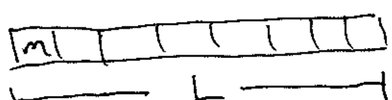
$$U(x) = \tfrac{1}{2} k (x-a)^2 \quad \text{and} \quad k = U''(a)$$

II A solid at rest is at stable equilib and can be modeled by



masses connected by massless springs representing interactions

Simple case:

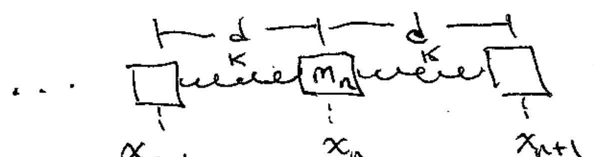


Rod, length L, mass M composed of N masses of mass $m = \frac{M}{N}$

Compress rod with force F,

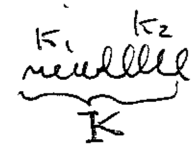
$$F = \mathbb{K}\,\Delta L$$

and $\mathbb{K}$ the effective spring constant of the whole rod. We model



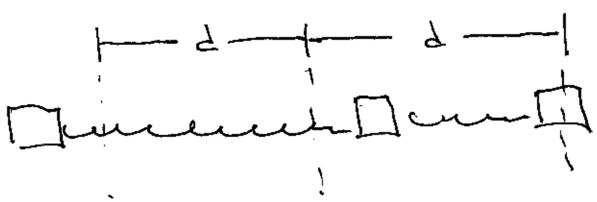
with $L = Nd$ and $\mathbb{K} = \frac{k}{N}$

Recall, for springs in series



$$1/k_1 + \frac{1}{k_2} = \frac{1}{\mathbb{K}}$$

Denote $\psi(x_n,t)$ the distance from equilibrium of the mass with eq. position x_n at time t. For example,



At $t=t_0$

$\psi(x_{n-1},t_0)<0$

$\psi(x_n,t_0)>0$

$\psi(x_{n+1},t_0)=0$

So, the force on m_n is:

Left: $F_L = -k[\psi(x_n,t)-\psi(x_{n-1},t)]$

Right: $F_R = k[\psi(x_{n+1},t)-\psi(x_n,t)]$

And again,

$$\frac{KL^2}{M}\frac{\partial^2\psi_n}{\partial x^2} = \frac{\partial^2\psi_n}{\partial t^2}.$$

This is the wave equation with speed

$$v = L\sqrt{\frac{K}{M}}$$

This simple model predicts the same speed for all waves. But, this is not what's observed $\rightarrow$ distinct ω's, k's have different v's ; unequal masses, spring constants, & the fact that $d \not\to 0$ leads to more complicated relations between frequency, wavelength and speed.

Total force:

$$F = k[\psi_{n+1}-\psi_n-(\psi_n-\psi_{n-1})] = m_n a$$

$$\Rightarrow \frac{d^2k}{m_n}\left[\frac{\frac{\psi_{n+1}-\psi_n}{d}-\frac{\psi_n-\psi_{n-1}}{d}}{d}\right] = \frac{\partial^2\psi_n}{\partial t^2}$$

Now,

$$\frac{kd^2}{m} = \frac{NK}{M/N}\left(\frac{L}{N}\right)^2 = \frac{KL^2}{M}$$

In the limit $d\to 0$ we get

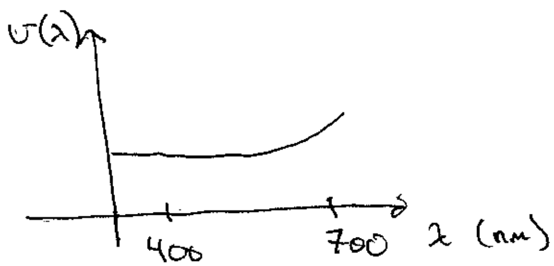
$$\frac{KL^2}{M}\lim_{d\to 0}\left[\frac{\frac{\partial\psi_{n+1}}{\partial x}-\frac{\partial\psi_{n-1}}{\partial x}}{d}\right] = \frac{\partial^2\psi_n}{\partial t^2}$$

In the general case we get a wave eq. w/ different speeds for different waves

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = (v(k))^2\frac{\partial^2\psi(x,t)}{\partial x^2}$$

[could write v as func. of $k, \lambda, \omega, \ldots$]

You've derive that for light in water

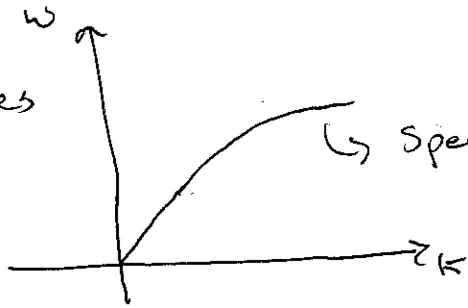


Red light is faster than blue; this is true of most materials for visible light and is called "normal dispersion".

We call $\omega(k)$ the dispersion relation

i.e long wavelength

Typical acoustic waves



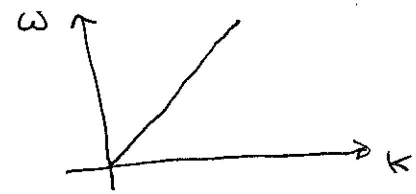
Harmonic soln.s still work for modified wave eqn : $A e^{i(kx - \omega(k)t)}$

III Combinations of these waves have a very rich set of behaviors.

Up to now we studied,

$$\omega(k) = v k$$

with v a constant — this is a linear dispersion relation



It generally holds for (i) light in vacuum and (ii) accoustic waves at low k,

Check out youtube videos

Group Velocity / Phase Velocity Anim.

stopped here

If we combine two waves with

$$\psi = A_1 e^{i(kx - \omega t)} + A_2 e^{i((k+\Delta k)x - (\omega + \Delta\omega)t)}$$

We find (as on your HW)

$$I \propto A_1^2 + A_2^2 + 2A_1A_2 \cos(\Delta k x - \Delta\omega t)$$

~~A~~ wave in its own right that moves with

$$v_g = \frac{\Delta\omega}{\Delta k} \xrightarrow{\text{limit}} \frac{d\omega}{dk}$$ "group velocity"