Modern
Day 4

Outline

I Lorentz transformations: velocity addition & the Spacetime invariant

II Four-Vectors

III Energy and Momentum in Relativity

I. Recall the Lorentz transformations!

$$x' = \gamma\left(x - \frac{v}{c}(ct)\right)$$

$$ct' = \gamma\left((ct) - \frac{v}{c}x\right)$$

Of course,

$$y' = y$$
$$z' = z$$

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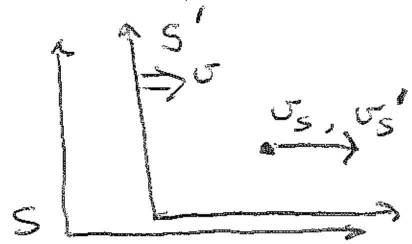
Velocity Addition



In S

$$\left.\begin{aligned}\Delta x &= x_2 - x_1 \\ \Delta t &= t_2 - t_1\end{aligned}\right\} \Rightarrow v_s = \frac{\Delta x}{\Delta t}$$

In S'

$$\left.\begin{aligned}\Delta x' &= x_2' - x_1' = \gamma(\Delta x - v\Delta t) \\ \Delta t' &= t_2' - t_1' = \gamma(\Delta t - \tfrac{v}{c^2}\Delta x)\end{aligned}\right\} v_s' = \frac{\Delta x'}{\Delta t'}$$

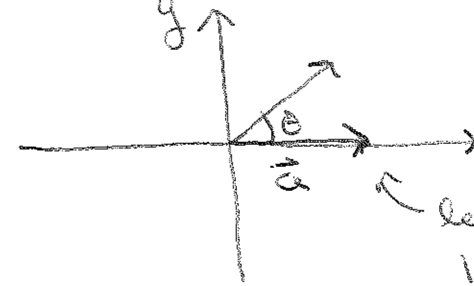
so,

$$v_s' = \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} = \frac{\frac{\Delta x}{\Delta t} - v}{1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}} = \boxed{\frac{v_s - v}{1 - \frac{v_s v}{c^2}}}$$

Spacetime Invariant

A very useful way to think about transformations is to ask what, if anything, they leave invariant.

Consider rotations



We have $\vec{a} = (1, 0)$ and

$$\vec{a}' = (\cos\theta, \sin\theta)$$

Geometrically it is clear that

$$|\vec{a}'|^2 = 1 \Rightarrow \cos^2\theta + \sin^2\theta = 1 \checkmark$$

Rotations leave the length of an (spatial) interval invariant. What about Lorentz transforms? Consider

$$(\Delta x')^2 - c^2 \Delta t'^2 = \gamma^2 \left[\Delta x^2 - 2v\Delta x\Delta t + v^2\Delta t^2 - c^2\left(\Delta t^2 - \frac{2v}{c^2}\Delta x\Delta t + \frac{v^2}{c^4}\Delta x^2\right)\right]$$

$$= \gamma^2\left[\left(1 - \frac{v^2}{c^2}\right)\Delta x^2 - (c^2 - v^2)\Delta t^2\right]$$

$$= \gamma^2\left[\frac{1}{\gamma^2}\Delta x^2 - \left(1 - \frac{v^2}{c^2}\right)c^2\Delta t^2\right]$$

$$= \Delta x^2 - c^2\Delta t^2$$

The interval

$$\Delta x^2 - c^2 \Delta t^2$$

is invariant, i.e. the same in *any* frame of reference.

Tidier Notation:

$x^0 \equiv ct, \quad x^1 \equiv x, \quad x^2 \equiv y, \quad x^3 \equiv z$

$$(x^0)' = \gamma (x^0 - \beta x^1), \quad \beta \equiv \frac{v}{c}$$

$$(x^1)' = \gamma (x^1 - \beta x^0)$$

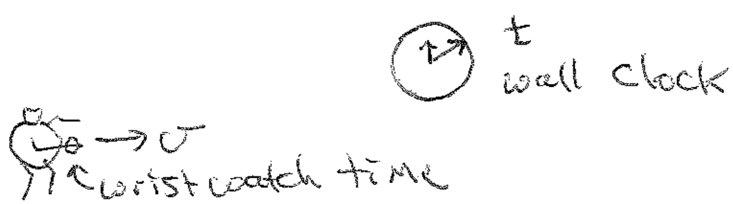
$$(x^2)' = x^2 ; \quad (x^3)' = x^3$$

Position-time 4-vector:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

Let's build up another example.

Proper time: Two times: "ordinary" ("coordinate") time. Shown on lab wall clock — t.



(Wrist watch time is) "proper" – shown on a particle's own watch: τ.

II Four-Vectors:

A "4-vector" is any collection of 4 #'s $a^\mu = (a^0, a^1, a^2, a^3)$ that transforms in the same way that x^μ does when you go from S to S'

$$(a^0)' = \gamma (a^0 - \beta a^1)$$

$$(a^1)' = \gamma (a^1 - \beta a^0)$$

$$(a^2)' = a^2 ; \quad (a^3)' = a^3 .$$

(Note: Proper time is same for all observers; t is not.)

$$\boxed{dt = \gamma \, d\tau}$$

Proper Velocity: "Ordinary" velocity: $v = \frac{\Delta x}{\Delta t}$ (both lab system)

"proper" velocity:

$$\eta = \frac{\Delta x}{\Delta \tau} \quad \begin{matrix}\text{(lab system)} \\ \text{(moving system)}\end{matrix}$$

Notice: $v = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{\gamma \Delta \tau} = \frac{1}{\gamma}\eta$.

Proper velocity 4-vector

$$\eta^\mu = \frac{dx^\mu}{d\tau}$$

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{d\tau} = \gamma c$$

$$\eta^1 = \frac{dx^1}{d\tau} = \frac{dx}{d\tau} = \gamma \frac{dx}{dt} = \gamma v_x$$

$$\eta^2 = \frac{dx^2}{d\tau} = \gamma v_y$$

III Momentum: classically: mass·velocity

$\vec{p} = m\vec{v}$, or $\vec{p} = m\vec{\eta}$?

↳ This def. would ⇒ that conservation of momentum is not even a possible law — if true in S, not true in S'.

Define: Relativistic momentum:

$$\vec{p} = m\vec{\eta} = \gamma m \vec{v} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

This means that $\vec{p}$ is the spatial

So,

$$\eta^\mu = (c\gamma, v_x\gamma, v_y\gamma, v_z\gamma)$$

$$= \gamma(c, v_x, v_y, v_z).$$

Note that

$$\gamma = \frac{1}{\sqrt{1 - \frac{1}{\gamma^2}\frac{\eta^2}{c^2}}} \Rightarrow \gamma^2 = \frac{1}{1 - \frac{1}{\gamma^2}\frac{\eta^2}{c^2}}$$

$$\Rightarrow \gamma^2\left(1 - \frac{1}{\gamma^2}\frac{\eta^2}{c^2}\right) = 1 \Rightarrow \gamma^2 = 1 + \frac{\eta^2}{c^2}$$

$$\Rightarrow \gamma = \sqrt{1 + \eta^2/c^2}.$$

part of a 4-vector:

$$p^\mu = m\eta^\mu$$

Question: What's the zeroth ("temporal") component?

$$p^0 = m\eta^0 = \gamma m c$$

Define: Relativistic energy

$$E = \gamma m c^2$$

or

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$