Homework 12

Due Friday, December 14th at 5pm

Problem 1 (Two body Hydrogen)
When we studied hydrogen we pretended that the nucleus did not move at all while the electron orbited it. How good of an approximation is this? This problem tackles that question. What if we treated a system with a lighter substitute for the nucleus, like positronium which is made up of an electron and a positron orbiting one another? Now that you have studied the quantum mechanics of two particles interacting you can answer these questions.

Often the potential energy of two particles only depends on the separation $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ of the particles, $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1 - \vec{r}_2)$. Inspired by classical mechanics we can make a change of variables from $\vec{r}_1$ and $\vec{r}_2$ to the separation vector $\vec{r}$ and the center of mass vector $\vec{R} \equiv (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$.

Define the reduced mass by $\mu \equiv (m_1 m_2)/(m_1 + m_2)$. Then it is not too hard to show that

$$\vec{r}_1 = \vec{R} + \frac{\mu}{m_1}\vec{r} \qquad \text{and} \qquad \vec{r}_2 = \vec{R} - \frac{\mu}{m_2}\vec{r},$$
$$\nabla_1 = \frac{\mu}{m_2}\nabla_R + \nabla_r \qquad \text{and} \qquad \nabla_2 = \frac{\mu}{m_1}\nabla_R - \nabla_r.$$

(You should convince yourself of these facts at some point, but you do not have to show this for this problem unless you want to.)

(a) Starting from the two particle Schrödinger equation

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V(\vec{r}_1, \vec{r}_2)\psi = E\psi,$$

show that the (time-independent) Schrödinger equation becomes

$$-\frac{\hbar^2}{2(m_1 + m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\vec{r})\psi = E\psi.$$

(b) Separate the variables, letting $\psi(\vec{R}, \vec{r}) = \psi_R(\vec{R})\psi_r(\vec{r})$. Note that ψ_R satisfies the one-particle Schrödinger equation, with the *total* mass $(m_1 + m_2)$ in place of m, potential zero, and energy E_R, while ψ_r satisfies the one-particle Schrödinger equation with the reduced mass in place of m, potential $V(\vec{r})$, and energy E_r. The total energy is the sum: $E = E_R + E_r$. What this tells us is that the center of mass moves like a free particle, and the relative motion (that is, the motion of particle 2 with respect to particle 1) is the same as if we had a *single* particle with the *reduced* mass, subject to the potential V. Exactly the same decomposition occurs in *classical* mechanics; it reduces the two-body problem to an equivalent one-body problem. (Be sure to explain your logic as well as the mathematics for this problem.)

(c) Find (to two significant digits) the percent error in the binding energy of hydrogen introduced by our use of m instead of μ.

Problem 2 (Exotic atoms)
(a) Using the Rydberg formula that you studied in lab and the formula that we derived for the Rydberg constant from the Bohr model of the atom, find the separation in wavelength between the red Balmer lines ($n = 3 \to n = 2$) for hydrogen and **deuterium**, which consists of a single electron orbiting a nucleus made up of a proton and a neutron.

Using your results from **Problem 1**:
(b) Find the binding energy of **positronium** (in which the proton is replaced by a positron—positrons have the same mass as electrons, but opposite charge).

(c) Suppose you wanted to confirm the existence of **muonic hydrogen**, in which the electron is replaced by a muon (same charge, but 206.77 times heavier). Where (i.e. at what wavelength) would you look for the "Lynman-α" line ($n = 2 \to n = 1$)?

Problem 3 (Normalization of identical particles)

(a) If ψ_a and ψ_b are orthogonal, and both normalized, what is the constant A in the equation

$$\psi_\pm(x_1, x_2) = A[\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)],$$

where the plus sign in the formula is used for **bosons** and the minus sign for **fermions**.

(b) If $\psi_a = \psi_b$ (and it is normalized), what is A? (This case, of course, occurs only for bosons.)

Problem 4 (Identical particles in a square well)
Imagine two noninteracting particles, each of mass m, in the infinite square well. If one is in the state ψ_n (Equation 2.28 of our excerpt), and the in state ψ_ℓ ($\ell \neq n$), calculate $\langle (x_1 - x_2)^2 \rangle$, assuming (a) they are distinguishable particles, (b) they are identical bosons, and (c) they are identical fermions.

Problem 5 (Electron configurations)
Figure out the electron configurations, in the notation $(1s)^2(2s)^2 \cdots$, for the first two rows of the Periodic Table (up to neon).