Homework 7

Due Friday, November 2nd at 5pm

Begin reading Chapter 2 of Schroeder's book *Thermal Physics.*

Problem 1 (Cooling a thermos)
An insulated Thermos contains 130 g of water at 80°C. You put in a 12 g ice cube at 0°C to form a system of *ice + original water*. (a) What is the equilibrium temperature of the system? What are the entropy changes of the water that was originally the ice cube (b) as it melts and (c) as it warms to the equilibrium temperature? (d) What is the entropy change of the original water as it cools to the equilibrium temperature? (e) What is the net entropy change of the *ice + original water* system as it reaches the equilibrium temperature?

For the remainder of the course we are going to be using the ideas behind probabilities more and more. To help get you oriented, I've added a short (7 page) reading on how to calculate averages and standard deviations using probabilities to our course site here.

Problem 2 (Age distribution)
For the distribution of ages in Section 1.3.1 of the new handout:
(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$.
(b) Determine Δj for each j, and use Eq. [1.11] to compute the standard deviation.
(c) Use your results in (a) and (b) to check Equation [1.12].

Problem 3 (Rock and cliff)
(a) Find the standard deviation of the distribution of Example 1.1 of the handout.
(b) What is the probability that a photograph, selected at random, would show a distance x more than one standard deviation away from the average?

Problem 4 (Multiplicity Example) In class we showed that the multiplicity of the binary model could be written as

$$\Omega(N,s) \approx \sqrt{\frac{2}{\pi N}} \frac{2^N}{(1-x^2)^{\frac{N+1}{2}}} \left(\frac{1-x}{1+x}\right)^s,$$

where you will recall that $N = N_L + N_R$ is the total number of particles that we put into the binary box, $s = N_L - N/2$, and $x = 2s/N$.
(a) Find the Taylor expansions of e^x, e^{-x}, and e^{-x^2}. Truncate these series at the first non-trivial power of x to find approximate expressions for the exponentials.
(b) For large N note that $N + 1 \approx N$ and use this approximation and those of part (a) to show that

$$\Omega(N,s) \approx \sqrt{\frac{2}{\pi N}} 2^N e^{-\frac{2s^2}{N}}.$$

(c) Find the probability distribution $P(N, s)$ of the macrostate with N particles and an imbalance s. In class Friday, I made a statement about this distribution that was just wrong. Use Python to plot this function for the three values $N = 10, 20, 100$. Does the distribution get narrower and more sharply peaked as N increases?

I told you that the distribution should get more sharply peaked as N increases. I made the error, because that is how most physical systems work. However, in our binary model it turns out to be a little more subtle. The rest of this problem set fixes my mistake and explains what actually happens. The tools you'll use to show it are very useful and your results will be quite nice too.

Problem 5 (Gaussian Integral)
In class on Monday we will show that

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

(a) Use substitution to prove that

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}.$$

This is the most useful integral you will encounter in your undergraduate career and probably for much longer. I highly recommend memorizing this result.
(b) Find the value of the integral

$$\int_{0}^{\infty} e^{-ax^2}dx.$$

[Hint: This value is best found without doing any further calculation.]
(c) Find the values of the integrals

$$\int_{-\infty}^{\infty} xe^{-ax^2}dx, \quad \int_{-\infty}^{\infty} x^3e^{-ax^2}dx, \quad \int_{-\infty}^{\infty} x^5e^{-ax^2}dx, \cdots .$$

[Hint: I'm asking you to do an infinite set of integrals, so again calculation would be ill advised.]
(d) Using your result from part (a), take the derivative of both sides of that equation with respect to a to find the value of the integral

$$\int_{-\infty}^{\infty} x^2e^{-ax^2}dx.$$

(e) Explain how you would find the integrals

$$\int_{-\infty}^{\infty} x^4e^{-ax^2}dx, \qquad \text{and} \qquad \int_{-\infty}^{\infty} x^6e^{-ax^2}dx.$$

Just words is enough for this part, you don't need to actually do the calculations.

Problem 6 (Gaussian practice)
Consider the **gaussian** distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where $A, a,$ and λ are positive real constants.
(a) Use Eq. [1.16] of the handout to determine A. We call this normalizing the distribution.
(b) Find $\langle x\rangle$, $\langle x^2\rangle$ and σ. What is the σ of your answer for problem 4(c)? Since σ characterizes the width of the gaussian, how does this fit with what you found about how the width of the distribution from Problem 4 changes with N?

Problem 7 (Final insight for the binary model)
(a) Use what you have learned about gaussians and your result for $P(N, s)$ from Problem 4 to compute the average imbalance $\langle s \rangle$ and the average squared imbalance $\langle s^2 \rangle$.
(b) The key to understanding the physically relevant prediction of this model is to consider the fractional root mean square imbalance

$$\frac{\sqrt{\langle s^2 \rangle}}{N}.$$

Why do we consider $\sqrt{\langle s^2 \rangle}$ instead of $\langle s \rangle$? Why is it physically important to divide by N? Having made those two arguments (in words!), compute $\sqrt{\langle s^2 \rangle}/N$ and describe what happens to it for large N.