# Homework 9

Due Wednesday, November 21st at 5pm

In this short problem set we will focus on deriving some of the facts that were known about black body radiation before Planck's argument directly from Planck's result.

**Problem 1** (The Stefan-Boltzmann Law)
Recall that the Stefan-Boltzmann law states that the power radiated from a black body of surface area $A$ is given by $P = \sigma A T^4$, where the Stefan-Boltzmann constant has the empirical value $\sigma = 5.67 \times 10^{-8}$ W/m$^2$ K$^4$.

(a) From our result for $u(T, \nu)$ argue that the combination

$$u(T, \nu)\, c\, d\nu,$$

with $d\nu$ a little differential interval of frequencies, has units of power per unit area. Comment on why multiplying $u$ by $c$ is the right way to get at power in this context.

(b) Argue that the integral

$$\int_0^\infty u(T, \nu)\, c\, d\nu$$

gives the total power radiated divided by the surface area of a black body of surface area $A$, that is, it gives $P/A$.

(c) By plugging in our result for $u(T, \nu)$ and changing variables to the dimensionless quantity $x = \frac{h\nu}{kT}$ show that $P \propto AT^4$ without explicitly doing the integral. Notice that this already allows you to comment on your measurement of the exponent in the Thermal Radiation Intensity lab even without carrying out the integration.

(d) Using the fact that

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15},$$

find an expression for the Stefan-Boltzmann constant $\sigma$ in terms of $\pi$, $c$, $k$, and $h$. Evaluate the value of your result and compare to the empirical value given above. Isn't it wonderful to see this come right out of the Planck result?

**Problem 2** (Wien's Law)
In class I also stated the Wien displacement law that the most energy is emitted at a wavelength $\lambda_{\max}$ that is inversely proportional to the temperature of the black body, that is,

$$\lambda_{\max} = \frac{\alpha}{T},$$

where the Wien constant $\alpha$ has the empirical value $\alpha = 2.8977 \times 10^{-3}$ K m.

(a) Using our result $u(T, \lambda)$ write out the equation that you would need to solve in order to find $\lambda_{\max}$. Don't try to solve it yet, but do simplify it as much as you can. Be careful in your derivation, you will need to apply both the product rule and the chain rule multiple times.

You should find that your result from part (a) can be written very neatly in terms of the dimensionless variable $x = \frac{hc}{\lambda kT}$. The easiest way to find $\lambda_{\text{max}}$ is first to solve your equation from part (a) for the appropriate value of $x$ and then to plug in the definition of $x$ and solve for $\lambda$. This is how we will proceed.

(b) Express your result from part (a) in terms of $x$. The resulting equation is transcendental and cannot be solved exactly in a straightforward manner. However, by properly organizing your equation you should be able to drop an exponentially small term and arrive at a very simple approximate value for $x$. Do this.

(c) Use your result from part (b) to find an equation for $\lambda_{\text{max}}$. Does your result scale properly with $T$?

(d) Find a formula and value for the Wien constant $\alpha$ using your approximation from (b). By what percentage does it differ from the empirical value?

(e) Find a better value for the Wien constant by using python. You can do this as follows: Start a new jupyter notebook, import the math package, and use the command `from scipy.optimize import fsolve` to import the fsolve command. The fsolve command allows you to numerically find where a function vanishes near a given starting point. Use your result from part (b) to define a function in python, call it fMax(x), that when it is set equal to zero will be an equation for the desired value of $x$. Then use the python command fsolve(fMax, InitGuess) to find the desired value of $x$. Here InitGuess is the value of your initial guess and it makes sense to use your approximate value for the solution from (b). [Hint: If you start with an equation of the form $g(x) = 5$, then you will want your fMax function to be defined by fMax$(x) = g(x) - 5$ since fsolve looks for **zeroes** of the function.]
Find a new value for the Wien constant using your new value for $x$. How does this new value compare to the empirical value?

In the last problem of the set, we will not directly derive a property of black bodies. However, this problem will be useful when we eventually do think about recovering Kirchhoff's law of thermal radiation from Planck's result.

**Problem 3** (Occupation of states)
(a) Based on Boltzmann's probability for occupying states and the Bohr model, in a gas of hydrogen at room temperature how many times more atoms have their electrons in the lowest energy orbit (known as the "ground state") than in the next lowest orbit (known as the "first excited state")? [Hint: You do not need to calculate the partition function $Z$ in order to answer this question.]

(b) What temperature would the gas have to be at for there to be 100 times as many atoms with electrons in the ground state as in the first excited state?