Modern
Day 5

Outline

I Last time

II Energy & Momentum in Relativity

III Conservation of relativistic energy/Mom.

IV Massless particles

I Lorentz transformations

$$\boxed{\begin{aligned} (x^0)' &= \gamma(x^0 - \beta x^1) \\ (x^1)' &= \gamma(x^1 - \beta x^0) \\ (x^2)' &= x^2 \,;\quad (x^3)' = x^3 \end{aligned}}$$

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

A four-vector a^μ is any collection of 4 #'s that transforms like x^μ does.

Proper time: particle's wrist watch time, call it τ.

$$dt = \gamma \, d\tau$$

Proper velocity

$$\eta = \frac{\Delta x}{\Delta \tau} = \gamma \frac{\Delta x}{\Delta t} = \gamma v$$

or more generally

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, v_x, v_y, v_z).$$

Note that

$$\gamma = \frac{1}{\sqrt{1 - \frac{1}{\gamma^2}\eta^2/c^2}} \Rightarrow \gamma^2 = \frac{1}{1 - \frac{1}{\gamma^2}\eta^2/c^2}$$

$$\Rightarrow \gamma^2\left(1 - \frac{1}{\gamma^2}\frac{\eta^2}{c^2}\right) = 1 \Rightarrow \gamma^2 = 1 + \eta^2/c^2$$

$$\Rightarrow \gamma = \sqrt{1 + \eta^2/c^2}$$

II Momentum: Classically: mass · velocity

So

$$\vec{p} = m\vec{v} \quad \text{or} \quad \vec{p} = m\vec{\eta} \; ?$$

The 1st def. would $\Rightarrow$ conservation of momentum is not a possible law – if true in S, not true in S'.

<u>Define</u>: Relativistic momentum:

$$\vec{p} = m\vec{\eta} = \gamma m \vec{v} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

This means that $\vec{p}$ is the spatial part of a 4-vector:

$$p^\mu = m\eta^\mu$$

Note: E is <u>not</u> zero for a particle at rest

$$E = mc^2 \equiv R \quad \text{"Rest energy"}$$

Then

$$T = E - R = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2 \quad \text{"Kinetic ener."}$$

$$= \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)mc^2 \approx \frac{1}{2}mv^2$$

(non-relativistic)

<u>Question</u>: What's the zeroth ("temporal") component?

$$p^0 = m\eta^0 = m\gamma c$$

<u>Define</u>: Relativistic energy

$$E = \gamma mc^2$$

or

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

So,

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) \quad \text{"Energy-Momentum 4-vector"}$$

<u>Useful Fact</u>:

$$E^2 - p^2c^2 = \frac{m^2c^4}{1 - v^2/c^2} - \frac{m^2v^2c^2}{1 - v^2/c^2}$$

$$= \frac{m^2c^4}{1 - v^2/c^2}\left(1 - \frac{v^2}{c^2}\right)$$

$$= m^2c^4$$

So, $\boxed{E^2 - p^2c^2 = m^2c^4}$

gets you directly from E to $\vec{p}$.

III Conservation Law:

$\boxed{\text{In any collision process (including decays) energy \& momentum are conserved}}$

Example: $m\,\bigcirc \xrightarrow{3/5\,c} \quad \xleftarrow{3/5\,c} \bigcirc\, m$ (before) $\qquad$ $\bigcirc$ (after)

Question: What's the mass of the composite lump?

(before) $\qquad$ (after)

$$\frac{2mc^2}{\sqrt{1-9/25}} = Mc^2$$

$$\Rightarrow M = \frac{2}{\sqrt{\frac{16}{25}}}m = \frac{10}{4}m = 2.5\,m$$

Mass is not conserved, in general!

Note: All forms of "internal" energy are included in the rest mass of a composite object ($m \sim E/c^2$)

E.g. $\pi^0 \to e^+ + e^-$

Rest energy and mass decrease, while K.E (kinetic energy) increases.

def. An elastic collision is one in which K.E. is conserved (in practice, same particles out as in).

IV $m = 0$

In classical mechanics: No such thing!

$p = mv = 0$, $T = \frac{1}{2}mv^2 = 0$, $F = ma = 0$ ✗ forget it!

Relativity: Loophole! If $v = c$, then the denominators are also zero:

$E = 0/0$? $p = 0/0$?

But,
$$E^2 - p^2c^2 = m^2c^4 = 0$$
$$\Rightarrow E^2 = p^2c^2$$
$$\Rightarrow E = |\vec{p}|c$$

In fact, they exist (photons, gluons, approximate for neutrinos)