Day 9
Modern

<u>Outline</u>

I. Last time

II. Complex numbers

III Complex waves

IV Intensity

I. We studied the wave equation

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = c^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

and found that the solutions were traveling waves

$$\psi = \psi(x \pm ct).$$

We decided to focus on <u>periodic</u>, harmonic waves of the form

$$\psi = A\cos(kx \pm \omega t + \phi).$$

wave number $k = \frac{2\pi}{\lambda}$ angular freq. $\omega = 2\pi f = \frac{2\pi}{\tau}$ phase

We also encountered Euler's formula

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

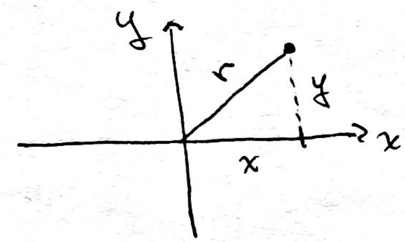
II Introduce the imaginary unit

$$i = \sqrt{-1}$$

so that

$$i^2 = -1.$$

In plane Euclidean geometry we have the pythagorean thm:



$$r^2 = x^2 + y^2$$

If we work with the

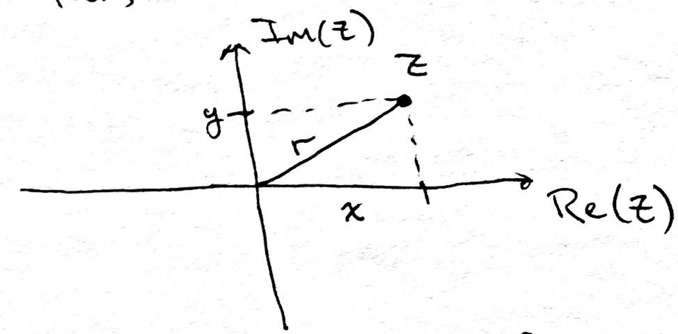
complex number $z = x + iy$, (with $x, y \in \mathbb{R}$) we can introduce its "real" part

$$\mathrm{Re}(z) \equiv x$$

and its "imaginary" part

$$\mathrm{Im}(z) \equiv y.$$

In other words the imaginary part is the coefficient of i in $z = x + iy$. We can plot complex numbers on a diagram like this



If we call the length of the diagonal r again we still have

$$r^2 = x^2 + y^2$$

A Complex number z is really just a convenient combination of two real numbers. If $r = 1$ then both the geometry and the identity $\cos^2\theta + \sin^2\theta = 1$ suggest we write

$$z = \cos\theta + i\sin\theta$$

Keep this in mind we'll return to it in a moment.

Notice that

$$\omega^2 = -1$$

has two solutions

$$\omega = \pm\sqrt{-1} = \pm i$$

This motivates introducing a new operation we call complex conjugation and denote with a $*$. This operation is defined by

$$i^* = -i$$

and when $z = x + iy$ so that

$$z^* = x - iy.$$

In general you can prove

that for any complex expression f the conjugate f^* has every i replaced by $-i$. So,

$$(e^{i\theta})^* = e^{-i\theta} = \frac{1}{e^{i\theta}}$$

This means that

$$e^{i\theta}\cdot(e^{i\theta})^* = e^{i\theta}e^{-i\theta} = e^{i\theta-i\theta}$$
$$= 1$$

In fact, in general we define the "magnitude" of a complex number as

$$|z|^2 = z\cdot z^*$$
$$= (x+iy)(x-iy)$$
$$= x^2+iyx-iyx-i^2y^2$$
$$= x^2+y^2 = r^2$$

complex multip. is just regular multip. with the identity $i^2=-1$

This is sensible. We just learned that

$$|e^{i\theta}|^2 = 1 = r^2$$

While not a proof this strongly motivates the Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Adding, subtracting, integrating, & differentiating complex numbers doesn't mix real and imaginary parts, e.g.

$$z = x+iy$$
$$\omega = u+iv$$
$$\omega+z = u+x+i(v+y).$$

This is not true for multiplying and dividing, e.g.,

$$\mathrm{Re}(\omega\cdot z) = \mathrm{Re}(xu-yv+i(yu+vx))$$
$$= xu-yv$$
$$\neq \mathrm{Re}(\omega)\cdot\mathrm{Re}(z) = xu.$$

III We're going to find it very useful algebraically to convert

$$A\cos(kx-\omega t+\phi) \to Ae^{i(kx-\omega t+\phi)}$$

Re

The expression on the right has

two harmonic waves encoded in it.
It will be useful to always work with both and pick out the one we're interested in

$$\mathrm{Re}\left(A e^{i(kx-\omega t+\phi)}\right).$$

This drops the imaginary piece, and in that sense loses some information, still it's surprisingly useful. However, if you need both parts you can always extract them using:

$$x = \mathrm{Re}(z) = (z + z^*)/2$$
$$= (x+iy+x-iy)/2$$
$$= x \checkmark$$

and

$$y = \mathrm{Im}(z) = (z - z^*)/2i$$
$$= (x+iy-x+iy)/2i$$
$$= y. \checkmark$$

IV complex numbers are going to be particularly convenient for doing calculations with intensity. What is intensity?

I like to think of the brightness of a star; it depends on the star's power $= \frac{\text{Energy}}{\text{time}}$, but also on how far away you are

← power spread out over sphere.

$$\boxed{\text{Intensity} \equiv \frac{\text{Energy}}{\text{Time}\cdot\text{Area}} = \frac{\text{Power}}{\text{Area}} \equiv I}$$

On the homework you will show that the energy of a wave is proportional to its square and so is its intensity

$$I \propto (\Psi(x,t))^2 = A^2\cos^2(kx-\omega t+\phi)$$

This is periodic with frequency 2ω.