# Homework 3

Due Thursday, September 24th at 10pm

Read the excerpt of Ch. 5 from Hartle's book. Read the remainder of Ch. 1, Secs. 1.7-12 in your Lyons text (pp 21-39).

**Problem 1** So far we have largely used spacetime diagrams to discuss events as described by an observer in just one frame, the $S$ frame. It will be very useful to be able to describe events from the perspective of two different frames on one and the same diagram. In particular, this is helpful for **Problem 2** below. To do this, let's imagine that frame $S'$ is moving with a constant speed $v > 0$ relative to $S$.

We know already that objects moving at a constant speed trace out a straight line with slope greater than 1 in a spacetime diagram. An object sitting still at the origin $x' = 0$ according to observers in $S'$ will sweep out such a line in the spacetime diagram of an $S$ observer. Not only that, but this object will report that in $S'$ all that is happening is that the time $t'$ is passing. Thus the $ct'$ axis must be a line in the $ct$-$x$ spacetime diagram with a slope $m$ such that $1 < m < \infty$.

(a) Find the equation of this line assuming that the two frames coincide at $t = t' = 0$. [Hints: Of course, a straight line always has the form $y = mx + b$ where $m$ is the slope of the line and $b$ is its $y$-intercept. Here you have to think of $y$ as $ct$ and $x$ as $x$. First write down the Lorentz transformation expressing $x'$ in terms of $x$ and $ct$. Next set $x' = 0$, since we are assuming the object is sitting at $x' = 0$. Finally, solve the resulting equation for the line by finding $ct$ in terms $x$. The equation you have just found is the equation for the $ct'$ axis expressed as a line in the $ct$ and $x$ spacetime diagram.]

(b) Taking $v = 3/5c$, draw a spacetime diagram as close to scale as possible that shows the $ct$, $x$, and $ct'$ axes.

(c) This time suppose we want to describe all events in $S'$ that happen at the time $t' = 0$. These events will sweep out the $x'$ axis. Repeat the steps of part (a), but for $t' = 0$ to find the equation of this line.

(d) Again with $v = 3/5c$, add the $x'$-axis to your diagram from part (b).

(e) How are the slopes of the $ct'$-axis and the $x'$-axis related in your $ct$-$x$ spacetime diagram? Does this result hold in general? That is, for any $v$? What does this mean about how these axes are situated around the light cone?

**Problem 2** A 20-m pole is carried so fast in the direction of its length that it appears to be only 10m long in the laboratory frame. The runner carries the pole through the front door of a barn 10 m long. Just at the instant the head of the pole reaches the closed rear door, the front door can be closed, enclosing the pole within the 10-m barn for an instant. The rear door opens and the runner goes through. From the runner's point of view, however, the pole is 20 m long and the barn is only 5 m! Thus the pole can never be enclosed in the barn. Explain, **quantitatively and by means of spacetime diagrams**, the apparent paradox.

**Problem 3** Particle $A$, at rest, decays into particles $B$ and $C$ ($A \to B + C$). [Hint: You will want to use both conservation laws and the spacetime invariant for the energy-momentum 4-vectors.]

(a) Find the energy of the outgoing particles, in terms of the various masses.

$$\left[ \textit{Answer:} \qquad E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} c^2 \right]$$

(b) Find the magnitudes of the outgoing momenta.

$$\left[ \begin{array}{l} \textit{Answer:} \qquad |\vec{p}_B| = |\vec{p}_C| = \dfrac{\sqrt{\lambda(m_A^2, m_B^2, m_C^2)}}{2m_A} c, \\ \quad \text{where } \lambda \text{ is the so-called } \textit{triangle function}: \\ \qquad \lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx. \end{array} \right]$$

(c) Note that $\lambda$ factors: $\lambda(a^2, b^2, c^2) = (a+b+c)(a+b-c)(a-b+c)(a-b-c)$. Thus $|\vec{p}_B|$ goes to zero when $m_A = m_B + m_C$, and runs imaginary if $m_A < (m_B + m_C)$. Explain.

**Problem 4** Consider a collision in which particle $A$ (with mass $m_A$ and proper velocity $\eta_A$) hits particle $B$ (mass $m_B$, proper velocity $\eta_B$), producing particle $C$ ($m_C$, $\eta_C$) and particle $D$ ($m_D$, $\eta_D$). Suppose that (relativistic) energy and momentum are conserved in system $S$ (i.e., $p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu$). Using the Lorentz transformations, show that (relativistic) energy and momentum are also conserved in $S'$. (Do not assume that mass is conserved—in general, it is not: $m_A + m_B \neq m_C + m_D$.)

**Problem 5** Starting next week in lab, it will be useful to have thought a little about random walks. Read through and complete the exercises in Python and Jupyter 3: Random Walks.

Once you understand how the random walk works, choose a number of times steps to run it for and compute the displacement $x$ of the walker at the end of this run. Repeat this 10 times, each time saving the total displacement of the walker at the end of the run. After these 10 runs, compute the mean of the square of the 10 displacements, that is, compute $\overline{x^2}$ for these 10 runs. Save that.

Now increase the number of time steps and repeat the whole process 10 mores times. Do this for 7 different numbers of time steps.

Finally, make a plot in Python of the mean squared displacement $\overline{x^2}$ vs. the number of times steps that you used for each of your runs. What do you notice about this plot? Can you think of a way to model the relationship? If so, fit your results to your model and report the parameters of your fit.

Repeat everything you've done here, but now instead of running your code 10 times for each of your chosen number of time steps, run it 1000 times for each of them. Of course, you will not want to do this by hand. Now the model you want to use should be clearer.