# Modern Physics
# Day 37

Today



## *I. Last Time*

Hal had an error in his notes. He apologizes profusely! Please correct in your notes (and Hal has corrected the slides). In the discussion of the Wien law the expression should have read

$$u(T,\lambda) = \frac{1}{\lambda^5}\phi(\lambda T) = \frac{1}{\lambda^5}\frac{A}{e^{\frac{b}{\lambda T}} - 1}$$

with a $1/\lambda^5$, *not* a $1/\lambda^3$. Our other results on the box of thermal radiation so far are: the equation of state

$$P = \frac{1}{3}\frac{U}{V} = \frac{1}{3}u(T),$$

and the Stefan-Boltzmann law

$$u(T) = \int_0^\infty u(T,\lambda)d\lambda = AT^4.$$

We began to consider radiation in thermal equilibrium with a reservoir at constant temperature $T$. Using what we had understood about entropy we showed that in these circumstances the probability of any state at fixed energy $E_s$ was given by

$$P(E_s) = \frac{e^{-\frac{E_s}{kT}}}{Z},$$

where we had argued that $Z$ is a normalization constant that makes this truly a probability, but we hadn't yet found a formula for $Z$. This is our first order of business today.

Our main goal for today will be to derive Planck's guess for the black body spectrum, which had the form specified above

$$u(T,\lambda) = \frac{1}{\lambda^5}\phi(\lambda T) = \frac{1}{\lambda^5}\frac{A}{e^{\frac{b}{\lambda T}} - 1}.$$

The argument is a lovely synthesis of all of the ideas that we have explored in the course: it touches on light as a particle (which we saw in relativity) with $E = pc$; we will use our understanding of light as a wave (which we explored in the waves portion of the course) that causes charged particles to oscillate; we will use all the tools in thermal physics that we have built up over the last several weeks; and finally, this argument opens up an exploration of Quantum Mechanics, which we will spend the last two weeks of the course exploring.

## II. Averages in Boltzmann's Approach

Let us return to our discovery that a system containing a thermal gas of photons in contact with a constant temperature reservoir can be described in terms of probabilities with

$$P(E_s) = \frac{e^{-\frac{E_s}{kT}}}{Z}.$$

Our first order of business is to find the constant $Z$. But, we know that if we add up over all possibilities the probability of that possibility that we must get 1, so

$$1 = \sum_s P(E_s) = \sum_s \frac{e^{-\frac{E_s}{kT}}}{Z}.$$

In the last sum, $Z$ is a constant and so we can pull it out of the sum to find

$$1 = \frac{1}{Z}\sum_s e^{-\frac{E_s}{kT}},$$

and solve this equation for $Z$, giving

$$Z = \sum_s e^{-\frac{E_s}{kT}}.$$

Of course, we can't always carry out the sum over all states (although we will be able to do so in today's example), but when we can, the resulting formula is astoundingly useful. In fact, an example of why these tools are useful presents itself immediately.

On last week's homework you studied the fact that we can use probabilities to express averages in a simple form. For example, to compute the average energy we can write

$$\overline{E} = \sum_s E_s P(s).$$

Using our probability results from above we find that we can write this average energy in the form

$$\overline{E} = \sum_s E_s \frac{e^{-\frac{E_s}{kT}}}{Z} = \frac{1}{Z}\sum_s E_s e^{-\frac{E_s}{kT}} = \frac{\sum_s E_s e^{-\frac{E_s}{kT}}}{\sum_s e^{-\frac{E_s}{kT}}}.$$

This formula shows that if we can figure out all the configurations of the photons in our box and their energies, then we can compute their average energy completely from scratch. It's not easy to figure out these states and energies, but it's well worth the effort.

## III. States for Light in a Box and Their Energies

Let us first consider the case of a one-dimensional box. As we have already discussed, we can think of the box as having reflecting walls. This leads to the waves that persist being standing waves.

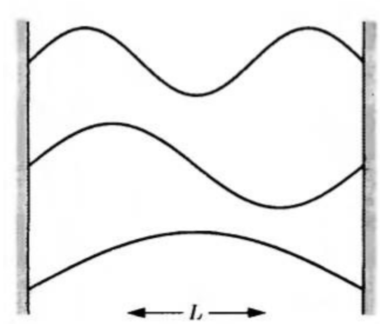


Figure 1: A 1D well with standing waves.

What are the wavelengths of these waves? We have

$$\lambda_1 = 2L = \frac{2L}{1}, \qquad \lambda_2 = \frac{2L}{2}, \qquad \lambda_3 = \frac{2L}{3}, \qquad \ldots.$$

In general, then

$$\lambda_n = \frac{2L}{n} \qquad \text{or} \qquad k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}.$$

In 3D, things get a bit more complicated because we can have different numbers of nodes (or antinodes) in each direction: $k_x = n_x\pi/L$, $k_y = n_y\pi/L$, and $k_z = n_z\pi/L$. So we have

$$\lambda_{(n_x,n_y,n_z)} = \frac{2\pi}{|\vec{k}|} = \frac{2\pi}{\sqrt{k_x^2 + k_y^2 + k_z^2}} = \frac{2L}{\sqrt{n_x^2 + n_y^2 + n_z^2}}.$$

Notice that at larger and larger $n_i$ there are more and more different wavelengths and hence different states of the light. We can estimate the total number of such states in a small interval of wavelength $d\lambda$ based on units

$$\#\ \text{states} \sim \frac{V}{\lambda^3}\frac{d\lambda}{\lambda}.$$

A slightly more rigorous treatment shows that there is a factor from the surface area of a sphere, $4\pi$, in this formula. Also, to account for all possible states we need to account for the fact that light can have two distinct polarizations. (Matt, I haven't discussed polarization with them at all.) Then, the total number of states per unit volume in this interval is

$$\frac{\#\ \text{states}}{V} = \frac{2\cdot 4\pi}{V}\frac{V}{\lambda^3}\frac{d\lambda}{\lambda} = \frac{8\pi}{\lambda^4}d\lambda.$$

This tells us how many states are contributing in each interval $d\lambda$ of wavelength. Next, we need the revolutionary proposal that Planck put forward.

Planck put forward that each of the allowed wavelengths could contribute a very particular energy to the total energy of the radiation. His proposal was that a constant with units distance times momentum, Planck's constant $h$, converted a standing wave's wavelength $\lambda$ to the energy, and that for each of these wavelengths there could be an integer number of such contributions

$$E_\lambda = \frac{hc}{\lambda}, \frac{2hc}{\lambda}, \cdots, \frac{nhc}{\lambda}.$$

This was a remarkable idea, that Planck wasn't sure exactly how to interpret. (This is closely related to Einstein's explanation of the photoelectric effect that you have been exploring in the lab and in modern parlance we would say that $n$ describes the "number of photons" in each "mode". Notice that Planck's proposal is equivalent to our laboratory discussion $E = nhf$ using $f = c/\lambda$.)

With this proposal in hand, we have everything we need to derive the black body spectrum.

## IV. The Spectral Energy Density of a Photon Gas

In our treatment, the spectral energy density

$$u(T,\lambda) = \frac{8\pi}{\lambda^4}\frac{\sum_{n=0}^{\infty}\frac{hnc}{\lambda}e^{-\frac{hnc}{\lambda kT}}}{\sum_{n=0}^{\infty}e^{-\frac{hnc}{\lambda kT}}}$$

is built from three pieces: (i) the first factor accounts for the number of states of the light in the box in the interval $d\lambda$ (The $d\lambda$ doesn't appear in this formula because, as you will recall, we are computing the "spectral" energy density, which is per unit wavelength; we will add it back in any time that we want to integrate over some interval of $\lambda$'s.), (ii) the first summand in the numerator, which gives the number of units of energy that are currently in that wavelength, and (iii) the probability of each of these states, which is encoded in the exponential summand in the numerator and in the full denominator. Notice the beautiful elegance of Boltzmann's procedure and Planck's implementation: there can be any number $n$ of energy units in a given "mode" with wavelength $\lambda$, but as the number of units gets large, the probability exponentially suppresses those states, which indeed should be very improbable. Delightfully we can compute everything in this formula!

Notice that the denominator is a geometric sum

$$\sum_{n=0}^{\infty}e^{-\frac{hnc}{\lambda kT}} = \sum_{n=0}^{\infty}\left(e^{-\frac{hc}{\lambda kT}}\right)^n = \sum_{n=0}^{\infty}\left(e^{-a}\right)^n = \frac{1}{1-e^{-a}} = \frac{1}{1-e^{-\frac{hc}{\lambda kT}}}.$$

While we can figure out the sum of the numerator, using the same trick that we used for the Gaussian integrals of your most recent homework. Taking a derivative of both sides of this result with respect to $a$ we find

$$\sum_{n=0}^{\infty} n\left(e^{-an}\right) = \frac{e^{-a}}{(1-e^{-a})^2}.$$

Putting these results together gives the result we were trying to derive(!)

$$u(T,\lambda) = \frac{8\pi hc}{\lambda^5}\frac{e^{-\frac{hc}{\lambda kT}}}{1-e^{-\frac{hc}{\lambda kT}}} = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1} = \frac{A}{e^{\frac{b}{\lambda T}}-1}.$$