Today

I. Note unusual homework due date/time; two more hw sets; take home due at end of completion days
II. Last Time
III. Statistical Mechanics and Equilibrium
IV. Boltzmann's Procedure
V. Kinetic Theory of a Gas of Photons

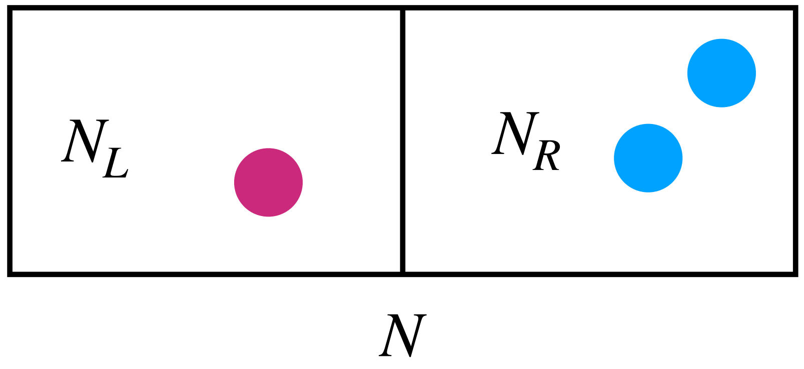
I. Last time we talked about microstates, macrostates, and multiplicity.

These definitions allow us to compute the entropy of the system:

$$S = k \ln \Omega.$$

II. When we put the particles out on a line we get $N!$ permutations of them.

We next select N_L particles to put on the left side of the box.



$$\Omega(N, N_L) = \frac{N!}{N_L!N_R!} = \frac{N!}{N_L!(N-N_L)!} \equiv \binom{N}{N_L}$$

$$\binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10.$$

$10^{23}!$ is so big that we can't really deal with it analytically.

<u>Suppose:</u> N is large, and the "imbalance" s is a small fraction of N

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We'll be using Stirling's approximation:

$$N! \approx \sqrt{2\pi N}\, e^{-N} N^N.$$

III. Applying these two approximations to our multiplicity $\Omega(N, N_L)$ we get

$$\Omega(N, N_L) \approx \sqrt{\frac{2}{\pi N}} \frac{2^N}{\left(e^{-x^2}\right)^{N/2}} \left(\frac{e^{-x}}{e^x}\right)^s = \sqrt{\frac{2}{\pi N}} 2^N e^{-2xs + \frac{N}{2}x^2} = \sqrt{\frac{2}{\pi N}} 2^N e^{-\frac{2s^2}{N}}$$

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If you study this distribution for larger and large N you find that it gets narrower and taller. What we're seeing is that as N gets large the fluctuations decrease. This is a generic property of thermal equilibrium and it is why we talk about "the equilibrium" of the system.

Notice that we can also compute the entropy of this binary model:

$$S = k \ln \left(\sqrt{\frac{2}{\pi N}} 2^N e^{-\frac{2s^2}{N}} \right).$$

IV. We're going to use all of these tools to study thermodynamics of all kinds of systems. Overview of how to use it:

Steps: Calculate the multiplicity for the system you care about Ω

Get Entropy: $S = k \ln \Omega$

Get Temperature: $\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_V$

Boltzmann Probability: $P(j) = \frac{e^{-E_j/kT}}{Z}$, j labels a particular state of your system and E_j is the energy of that state. Here Z is called the partition function and is defined by a sum over all states $Z = \sum_j e^{-E_j/kT}$ and guarantees that $\sum_j P(j) = 1$.

IV. Let's prove one of these steps:

Get Temperature: $\dfrac{1}{T} = \left(\dfrac{\partial S}{\partial U}\right)_V$

The 1st law states

$dQ = dU - dW = dU + PdV,$

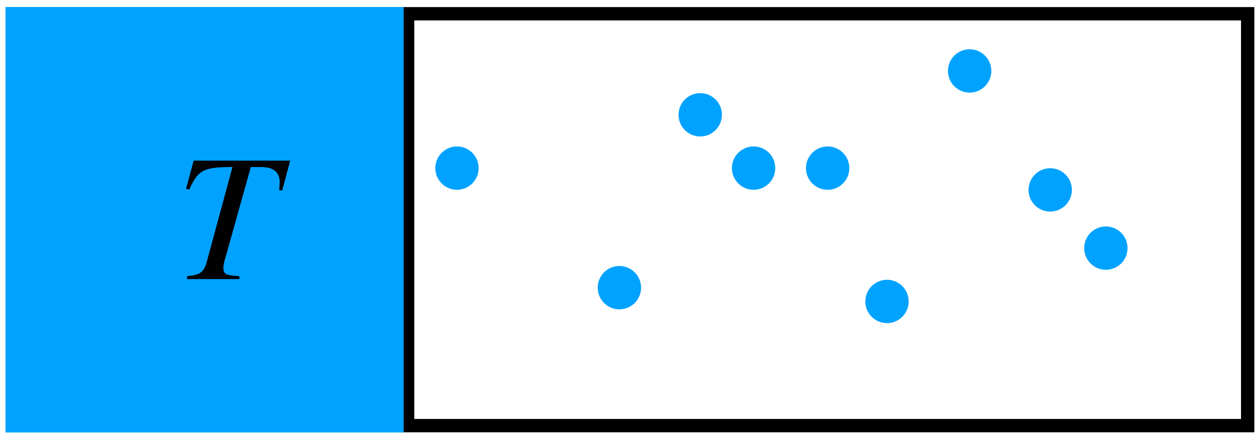
And dividing through by the temperature we have

$$\frac{dQ}{T} = \frac{1}{T}dU + \frac{P}{T}dV \quad \text{or} \quad dS = \frac{1}{T}dU + \frac{P}{T}dV.$$

We can find $(\partial S/\partial U)$ by holding the volume fixed and dividing by dU,

$$\left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T} \quad \text{or } T = \frac{1}{\left(\frac{\partial S}{\partial U}\right)_V}.$$

IV. Returning to the main theme of our course, let's consider a gas of photons in a box at temperature T.



"A box of light at temperature T". It will be convenient to introduce the energy density $u(T) = \dfrac{U}{V}$.

Light has energy, but also momentum. For light with energy E we have $p = E/c$. Since they have momentum, the light particles exert a force on the walls of the box, which leads to a pressure.

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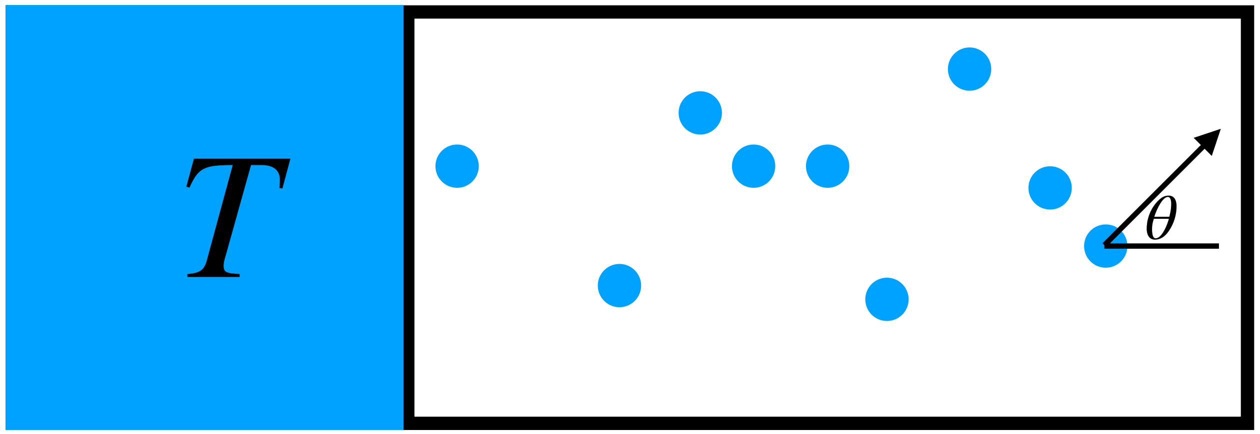

Consider a cubical box with side length L and hence volume $V = L^3$, then radiation at a constant temperature T, will also have $u(T)$ constant. The speed of a photon at angle θ is $c\cos\theta$, and it will hit the right wall every $2L/(c\cos\theta)$ seconds.