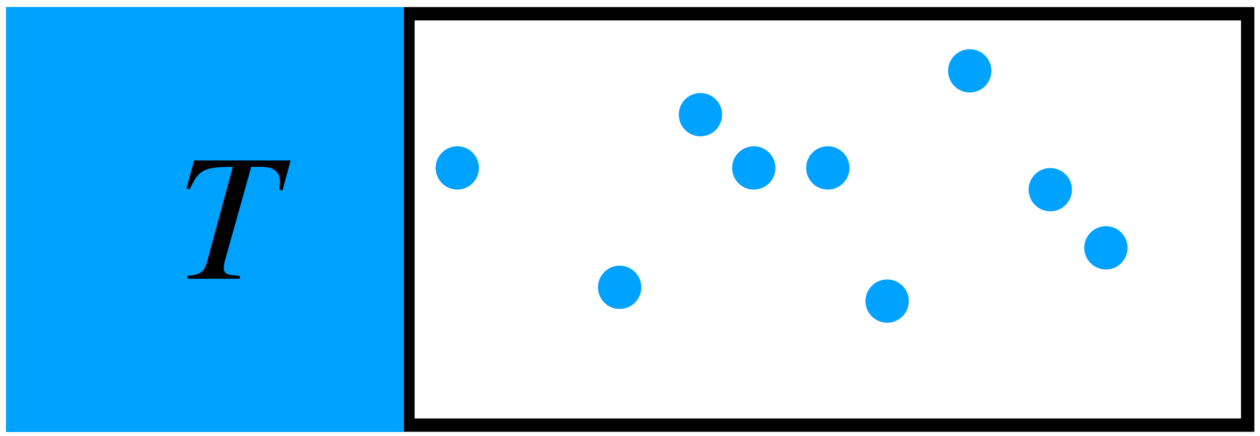
Today

I. Last Time
II. Spherical Coordinates and Spherical Integration
III. Kinetic Theory of a Gas of Photons
IV. The Power in a Gas of Photons

I. See below

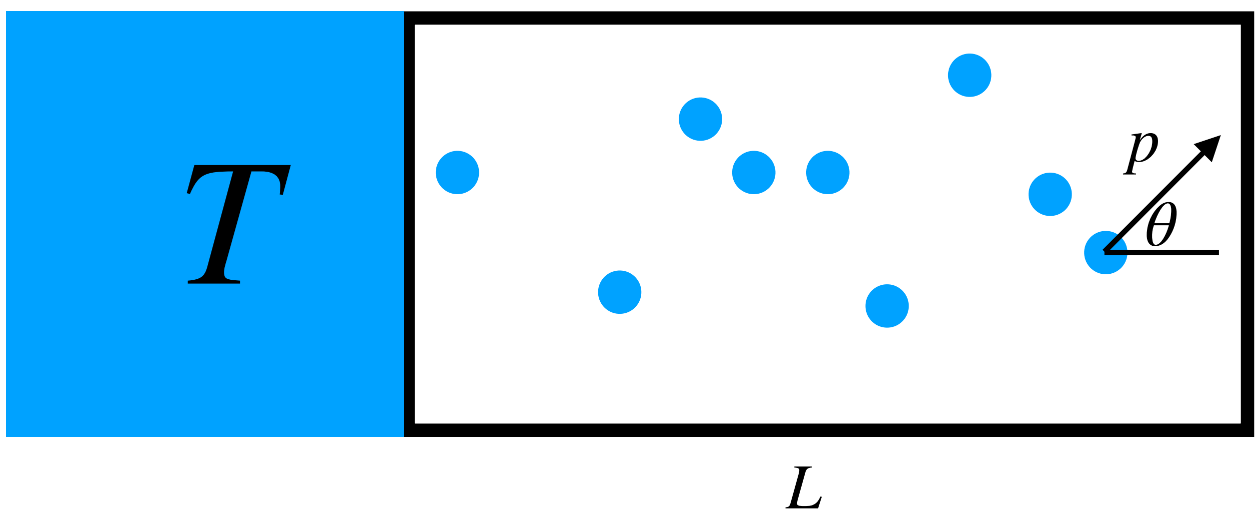
IV. Returning to the main theme of our course, let's consider a gas of photons in a box at temperature T.



"A box of light at temperature T". It will be convenient to introduce the energy density $u(T) = \dfrac{U}{V}$.

Light has energy, but also momentum. For light with energy E we have $p = E/c$. Since they have momentum, the light particles exert a force on the walls of the box, which leads to a pressure.

Light has energy, but also momentum. For light with energy E we have $p = E/c$. Since they have momentum, the light particles exert a force on the walls of the box, which leads to a pressure.



Consider a cubical box with side length L and hence volume $V = L^3$, then radiation at a constant temperature T, will also have $u(T)$ constant. The speed of a photon at angle θ is $c\cos\theta$, and it will hit the right wall every $2L/(c\cos\theta)$ seconds.

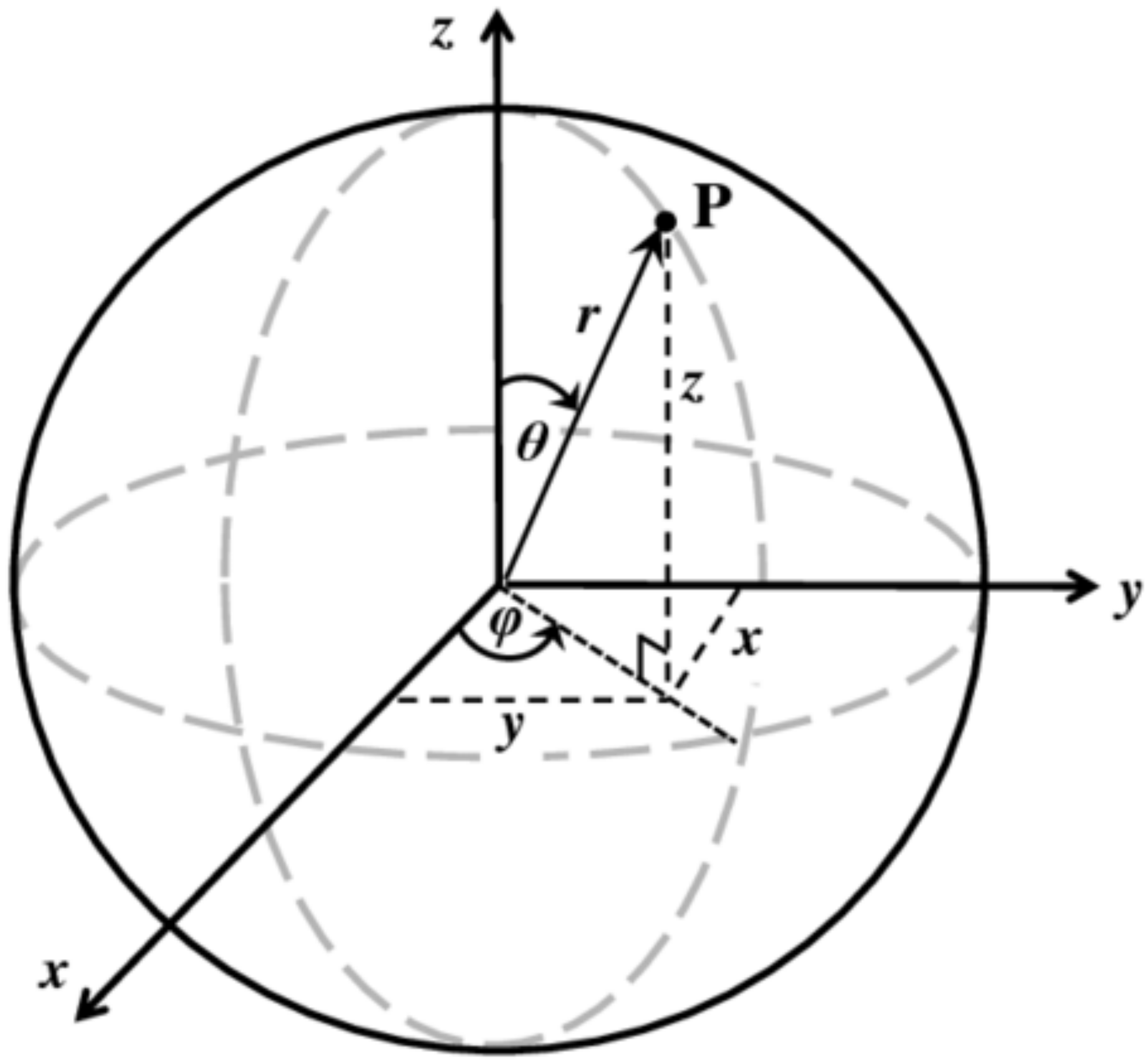
$$F = \frac{\Delta p}{\Delta t} = \frac{2p\cos\theta}{2L/(c\cos\theta)} = \frac{pc\cos^2\theta}{L} = \frac{E\cos^2\theta}{L};$$

$$P = \frac{Force}{Area} = \frac{E\cos^2\theta/L}{L^2} = \frac{E}{V}\cos^2\theta \quad \text{(for 1 photon)}$$

II. Spherical Coordinates

Polar angle: θ

Azimuthal angles: φ, ϕ



$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

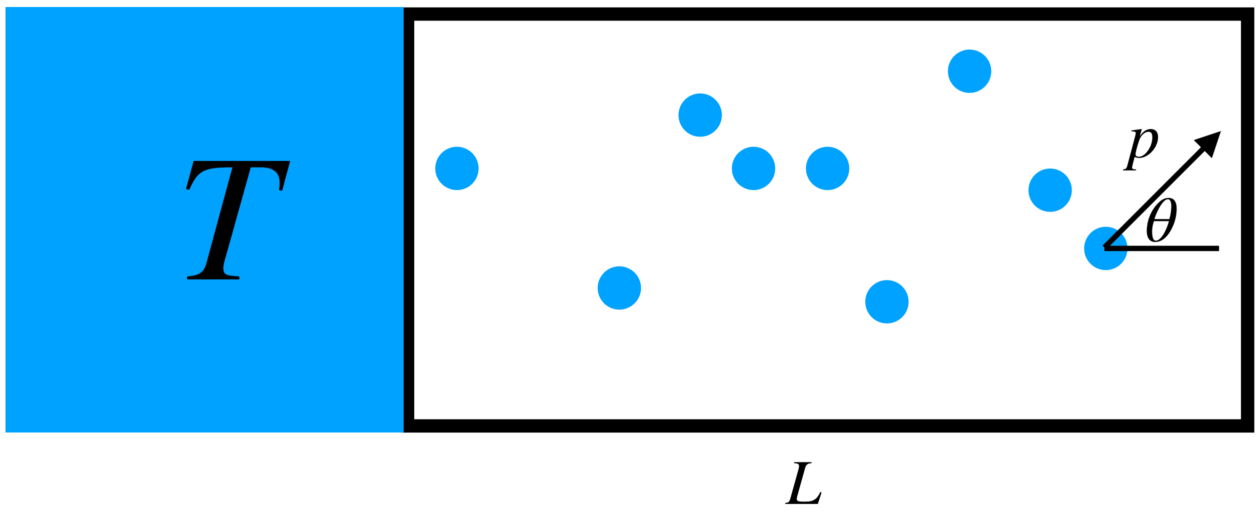
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}(z/r)$$

$$\varphi = \tan^{-1}(y/x)$$

Area Hemisphere

$$= \int_0^{2\pi} \int_0^{\pi/2} R^2 \sin\theta d\theta d\phi = R^2 \int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi/2} d\phi = R^2 \int_0^{2\pi} 1 d\phi = 2\pi R^2$$



In general, we will get photons contributing in every direction

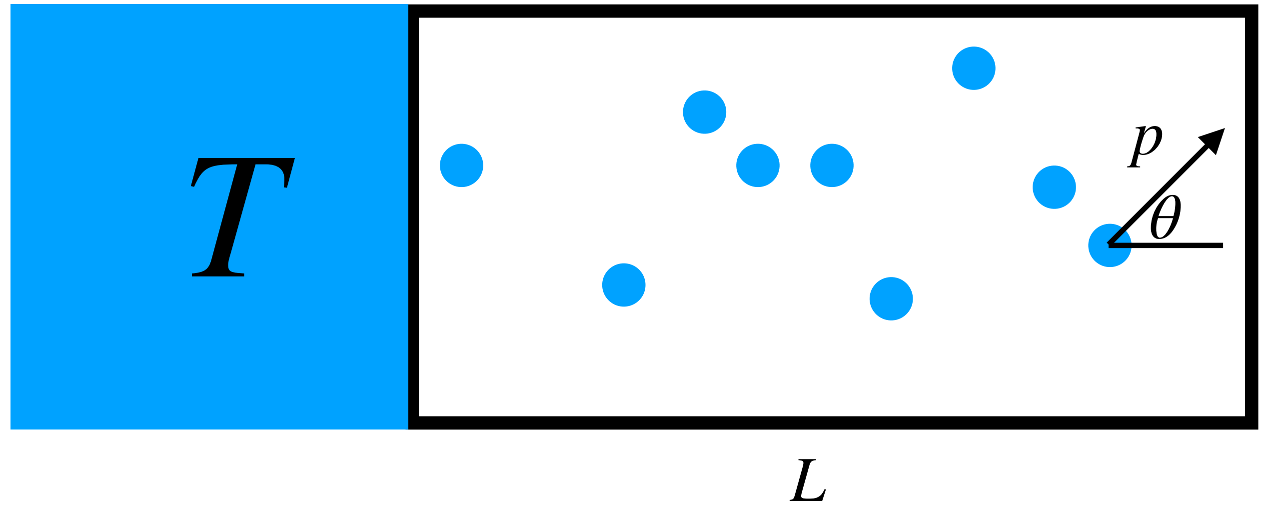
$$F = \frac{\Delta p}{\Delta t} = \frac{2p\cos\theta}{2L/(c\cos\theta)} = \frac{pc\cos^2\theta}{L} = \frac{E\cos^2\theta}{L};$$

$$P = \frac{Force}{Area} = \frac{E\cos^2\theta/L}{L^2} = \frac{E}{V}\cos^2\theta \ \text{(for 1 photon)}$$

We can find the total pressure by integrating up the contributions

$$P_{tot} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\frac{E}{V}\cos^2\theta\sin\theta d\theta d\phi = \frac{E}{V}\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta$$

$$= -\frac{1}{3}\left[\cos^3\theta\right]_0^{\pi/2}\frac{E}{V} = \frac{1}{3}\frac{E}{V} = \frac{1}{3}u$$



Switch from P_{tot} to just P and from E to U:

$$P = \frac{1}{3}\frac{U}{V} = \frac{1}{3}u.$$

Let's compare this to what we learned for the ideal gas

$$U = \frac{3}{2}NkT,$$

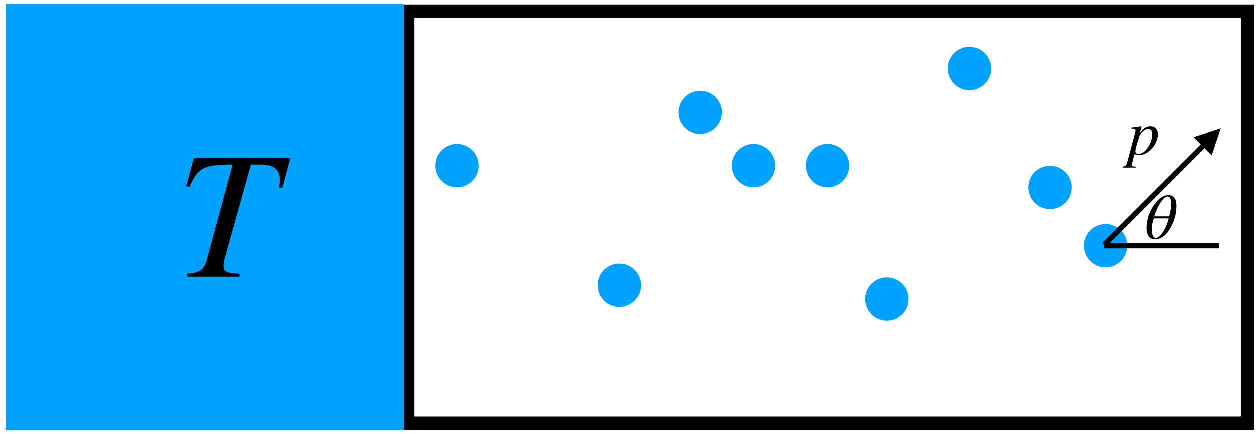
$$PV = NkT.$$

Then

$$U = \frac{3}{2}PV \qquad \text{or} \qquad P = \frac{2}{3}\frac{U}{V} = \frac{2}{3}u.$$

Today I want to use without proof the Gibbs-Duhem relation:

$$Ud\left(\frac{1}{T}\right) + Vd\left(\frac{P}{T}\right) = 0.$$



Switch from P_{tot} to just P and from E to U:

$$P = \frac{1}{3}\frac{U}{V} = \frac{1}{3}u.$$

Today I want to use without proof the Gibbs-Duhem relation:

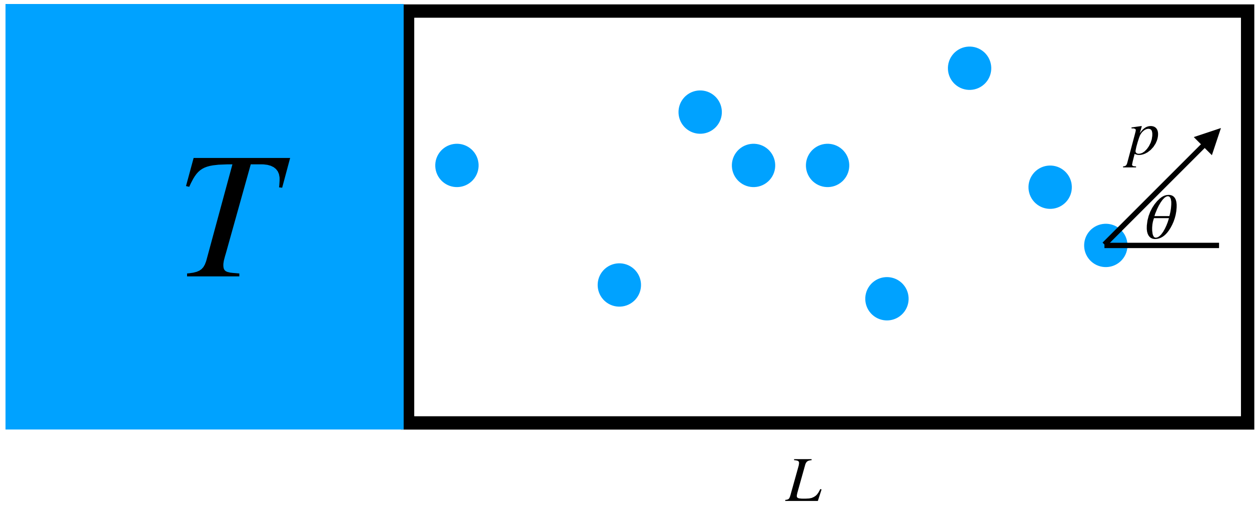
$$Ud\left(\frac{1}{T}\right) + Vd\left(\frac{P}{T}\right) = 0.$$

Divide everything by the volume

$$\frac{U}{V}d\left(\frac{1}{T}\right) + d\left(\frac{P}{T}\right) = 0 \implies 3Pd\left(\frac{1}{T}\right) + d\left(\frac{P}{T}\right) = 0,$$

Next divide both sides by P/T to get

$$\frac{3}{1/T}d\left(\frac{1}{T}\right) + \frac{1}{P/T}d\left(\frac{P}{T}\right) = 0.$$



Next divide both sides by P/T to get

$$\frac{3}{1/T}d\left(\frac{1}{T}\right) + \frac{1}{P/T}d\left(\frac{P}{T}\right) = 0.$$

Move one term to the other side to get

$$\frac{1}{P/T}d\left(\frac{P}{T}\right) = -\frac{3}{1/T}d\left(\frac{1}{T}\right)$$

And integrate both sides

$$\ln\left(\frac{P}{T}\right) = -3\ln\left(\frac{1}{T}\right) + \text{const} = \ln\left(T^3\right) + \text{const}.$$

Exponentiating both sides gives

$$\frac{P}{T} = aT^3 \qquad \text{or} \qquad P = aT^4 \qquad \text{or} \qquad u = \sigma T^4.$$