Today

I. No class on Wednesday (Advising days). There will be (optional) class Monday, Dec 14th.

II. Last Time

III. Wave-Particle Equations and Schroedinger

IV. The Infinite Square Well

II. Last time we discussed Bohr's model of the atom:

We used classical centripetal acceleration, Newton's 2nd law, Coulomb's law. We also used two important quantum inputs: De Broglie's $p = \frac{h}{\lambda}$, Bohr's input: $2\pi r = n\lambda$, $n = 1,2,3,\cdots$

$$E_n = \frac{E_1}{n^2}, \quad E_1 = -13.6 \text{ eV}, n = 1,2,3,\cdots$$

II. Today we will continue to draw inspiration from the wave-particle duality that Bohr was so interested in.

Return to our study of waves

$$E(x,t) = E_0 e^{i(kx-\omega t)}, \text{ here } k = \frac{2\pi}{\lambda}, \ \omega = \frac{2\pi}{T} = 2\pi \overset{\overset{v}{\swarrow}}{f}.$$

Notice the temptation to think of de Broglie

$$hk = 2\pi \frac{h}{\lambda} = 2\pi p \ \text{ or } p = \frac{h}{2\pi} k = \hbar k.$$

From Einstein we also have $E = \hbar\omega$. Let's write a completely general plane wave in the form

$$\Psi_{pl}(x,t) = A e^{i(\frac{xp}{\hbar} - \frac{Et}{\hbar})} = A e^{\frac{i}{\hbar}(xp - Et)}.$$

Let's probe this wave with an "operator" $\hat{p} \equiv \frac{\hbar}{i}\frac{\partial}{\partial x}$:

$$\hat{p}\Psi_{pl}(x,t) = \hat{p} A e^{\frac{i}{\hbar}(xp-Et)} = \frac{\hbar}{i}\frac{\partial}{\partial x}\left(A e^{\frac{i}{\hbar}(xp-Et)}\right) = \frac{\hbar}{i} A e^{\frac{i}{\hbar}(xp-Et)} \cdot \frac{i}{\hbar} p = p\Psi_{pl}(x,t)$$

II. Let's write a completely general plane wave in the form
$\Psi_{pl}(x,t) = Ae^{i(\frac{xp}{\hbar}-\frac{Et}{\hbar})} = Ae^{\frac{i}{\hbar}(xp-Et)}$.

Let's probe this wave with an "operator" $\hat{H} \equiv i\hbar\frac{\partial}{\partial t}$:

$\hat{H}\Psi_{pl}(x,t) = \hat{H}Ae^{\frac{i}{\hbar}(xp-Et)} = i\hbar\frac{\partial}{\partial t}\left(Ae^{\frac{i}{\hbar}(xp-Et)}\right) = i\hbar\left(-i\frac{E}{\hbar}\right)\Psi_{pl}(x,t) = E\Psi_{pl}(x,t)$

We call this the "Hamiltonian" operator, but we see here that it is an "energy" operator.

Schroedinger said the following: classically we have
$H = E = \frac{1}{2}mv^2 + V(x) = \frac{p^2}{2m} + V(x)$.

Schroedinger suggests that we write down the operator equivalent of this equation:

$$\hat{H}\Psi_{pl}(x,t) = \frac{\hat{p}^2}{2m}\Psi_{pl}(x,t) + \hat{V}(x)\Psi_{pl}(x,t).$$

II. A big leap! Sch. says this should hold for every quantum wave:

$$\hat{H}\Psi(x,t) = \frac{\hat{p}^2}{2m}\Psi(x,t) + \hat{V}(x)\Psi(x,t).$$

I also need to tell you what the potential energy operator is:

$$\hat{x}\Psi(x,t) = x \cdot \Psi(x,t).$$

With this in had we can define

$$\widehat{V(x)}\,\Psi(x,t) = V(x)\Psi(x,t)$$

With all the operators defined we can write out a differential equation, the Schroedinger equation,

$$\boxed{i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t).}$$

We are going to use separation of variables to attach this equation. Guess that we are interested in solutions of the form

$$\Psi(x,t) = \psi(x)\varphi(t).$$

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Putting this guess into the Sch. equation we have

$$i\hbar\psi\frac{d\varphi}{dt} = -\frac{\hbar^2}{2m}\varphi\frac{d^2\psi}{dx^2} + V\psi\varphi.$$

Divide both sides by $\varphi\psi$,

$$i\hbar\frac{1}{\varphi(t)}\frac{d\varphi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi}{dx^2} + V(x).$$

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From this equation, which sets $f(t) = g(x)$, we see that neither the left or the right hand side can depend on their respective variables! Then

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi}{dx^2} + V(x) = \text{const.} = E$$

or

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) \quad \text{(time-independent Sch. Eq.)}$$

We also have

$$i\hbar\frac{1}{\varphi(t)}\frac{d\varphi}{dt} = E \text{ or } i\hbar\frac{d\varphi}{\varphi} = Edt \text{ or } \ln\varphi = -i\frac{Et}{\hbar} + \text{const or } \varphi = Ae^{-\frac{iEt}{\hbar}}$$