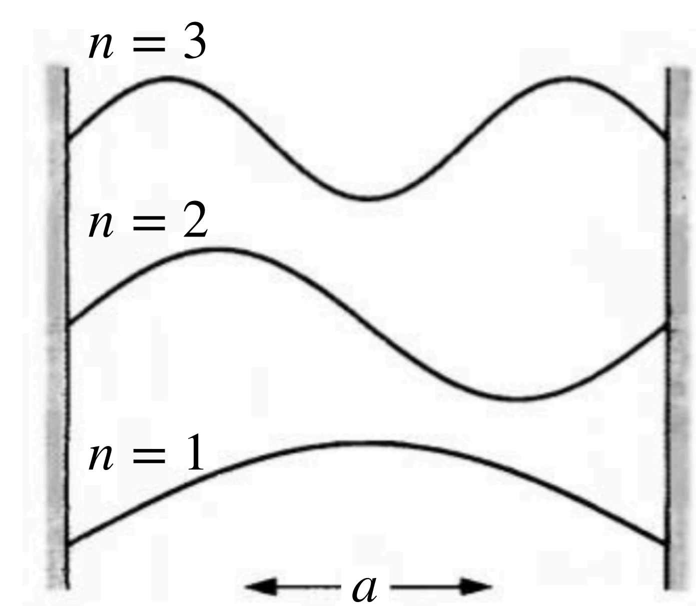
Today

I. Last Time
II. Normalization
III. The Probability Interpretation
IV. General Solutions to the Sch. Eq.

I. We were studying the infinite square well, which is a free particle confined within a box, with length a.

$$\psi(x) = A\sin\left(\frac{n\pi}{a}x\right),\ n = 1,2,3,\cdots.$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$



II. The "orthodox" (or Copenhagen) interpretation of quantum mechanics:

the wave function is a probability density

$|\psi(x)|^2 dx =$ the probability for finding the particle between x and $x + dx$

In other words, you integrate the absolute square of the wave function to find finite probabilities

$\int_{x=0}^{x=\frac{3}{4}a} |\psi(x)|^2 dx =$ probability of finding the particle between $x = 0$ and $x = \frac{3}{4}a$.

What does $\int_{x=0}^{x=a} |\psi(x)|^2 dx = ?$ Well, it's the total probability and hence it must be equal to 1.

II.What does $\int_{x=0}^{x=a} |\psi(x)|^2 dx = ?$ Well, it's the total probability and hence it must be equal to 1. We call the requirement

$$\int_{x=0}^{x=a} |\psi(x)|^2 dx = 1$$

the "normalization" requirement.

Let's do the example of our square well. First the absolute square $|\psi(x)|^2 = |A \sin\left(\frac{n\pi}{a}x\right)|^2 = A^2 \sin^2\left(\frac{n\pi}{a}x\right)$. Then,

$$\int_0^a A^2 \sin^2(kx)dx = A^2 \int_0^a \sin^2(kx)dx = A^2\frac{a}{2} = 1,$$

gives us that $A = \sqrt{\frac{2}{a}}$. Our total waves functions are now

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right), n = 1,2,3,\cdots.$$

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As mentioned above, to interpret this result we take its absolute square and integrate over some interval and say that is a probability. But, a probability of what?
It's the probability of measuring the position of the particle in the interval that you prescribed. Where was the particle before the measurement? We don't know and it may not be meaningful to talk about that.

Instead, what we have been focusing on is the repeated setup and carrying out of an experiment and the statistical agreement between our measurements and the predictions of quantum theory.

IV. Recall the full path of how we got here

1. The time-dependent Sch. Eq.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi.$$

2. Guess that the answer has the form $\Psi(x,t) = \psi(x)\varphi(t)$.

3. Solved for $\varphi(t) = e^{-iEt/\hbar}$.

4. Solve the time-independent Sch. Eq.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi.$$

5. For the infinite square well we just found

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right), \text{ n=1,2,3,....}$$

6. Put ‘em together

$$\Psi_n(x,t) = \psi_n(x)\varphi(t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-iEt/\hbar}$$

IV. How do we interpret

$$\Psi_n(x,t) = \psi_n(x)\varphi(t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-iEt/\hbar}?$$

Same way. We have

$|\Psi(x,t)|^2 dx =$ probability of finding the particle between x and $x + dx$ at the time t.

There's something neat to notice about taking the absolute square here:

$$|\Psi_n|^2 = \Psi_n\Psi_n^* = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-iEt/\hbar}\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{iEt/\hbar}$$

$$= \frac{2}{a}\sin^2\left(\frac{n\pi}{a}x\right)$$

This doesn't depend on time! We call these special states "stationary states". Surprisingly(!), you can build any solution out of these, by adding them up with different n.