1 Some Mathematical Preliminaries

There is some messy-ish math that comes up in this derivation. Rather than pause in the logical flow when the steps come up, I will collect these ideas first and refer to them when I need them.

1.1 Geometric Sums

You probably learned this in *Math Methods*, but it is so useful that I repeat it in some detail. Consider the sum of a series of terms that are just consecutive powers of the same base, y,

$$S_N = 1 + y + y^2 + \ldots + y^{N-1} = \sum_{n=0}^{N-1} y^n \; . \tag{1}$$

We can evaluate this by a cute multiplication-and-subtraction trick,

$$\begin{aligned} & S_N = 1 + y + y^2 + \ldots + y^{N-1} \\ \text{multiply by } y: \quad & y \cdot S_N = \quad y + y^2 + \ldots + y^{N-1} + y^N \\ \text{subtract, most terms cancel} \quad & (1-y) \cdot S_N = 1 - y^N \quad \Longrightarrow \quad S_N = \frac{1-y^N}{1-y} \; . \end{aligned} \tag{2}$$

Of special interest to us is the case when $|\, y \,| < 1$,

$$\begin{aligned} & \text{for} \quad |\, y \,| < 1 \, , \quad y^N \to 0 \;\; \text{as} \;\; N \to \infty \, , \text{so,} \\ & S = \lim_{N\to\infty} S_N = \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} \; . \end{aligned} \tag{3}$$

I can get a second infinite series sum by taking the derivative of S WRT y,

$$\begin{aligned} \text{take derivative of both sides of Eqn. (3),} \quad \frac{dS}{dy}: \quad & \sum_{n=0}^{\infty} n \cdot y^{n-1} = \frac{1}{(1-y)^2} \\ \text{multiply by } y: \quad y + 2\,y^2 + 3\,y^3 + \cdots = & \sum_{n=0}^{\infty} n \cdot y^n = \frac{y}{(1-y)^2} \; . \end{aligned} \tag{4}$$

1.2 Energies and exponents

I want to give advice to simplify your life when dealing with the myriad of formulas that you encounter in *Modern Physics*: **Get variables into units of energy when you can**.

All of these quantities have units of energy,

$$\text{Thermal Energy:} \quad k\,T$$

$$\text{Photon Energy;} \quad h\,f = \hbar\,\omega = \hbar\,k = \frac{h\,c}{\lambda}\,.$$

So, the quantity $\frac{h\,f}{k\,T} = \frac{h\,c}{\lambda\,k\,T}$ is dimensionless, and so is a suitable exponent for e.

You have learned about the Boltzmann factor that determines the relative likelihood for the population of different energy states. That is, given two possible energies for a system, E_1 & E_2, their populations have the ratio,

$$P(E) \,\propto\, e^{-E\,/\,kT} \quad\Longrightarrow\quad \frac{P(E_1)}{P(E_2)} = \frac{e^{-E_1\,/\,kT}}{e^{-E_2\,/\,kT}} \;=\; e^{-(E_1-E_2)\,/\,kT}\,. \tag{5}$$

This captures two essential ideas:

. (1) higher energy states are less likely to be populated (no matter what T is), and

. (2) as T increases, populating higher energy states becomes easier.

1.3 Switching from λ to f

When dealing with an energy density, you will always be measuring it over a finite range of frequencies or wavelengths, a df or $d\lambda$. Think of it as the range of light that the aperture on the spectrometer allows in. The energy density expression can be written in terms of either f or λ, but switching between them includes a non-intuitive factor. But, paying attention to units saves us. I give the two u expressions subscripts to be super-clear, but generally you call the density u and pay attention to argument to tell which variable is in play:

$$\begin{aligned}
\text{Total energy:} \quad & [\,U\,] = Energy\,.\\
\text{Energy per unit volume:} \quad & [\,u(T)\,] = Energy\,/\,Volume\,.\\
\text{Energy/Volume within } df: \quad & [\,u_f(T,\,f)\cdot df\,] = Energy\,/\,Volume\,.\\
\text{Energy/Volume within } d\lambda: \quad & [\,u_\lambda(T,\,\lambda)\cdot d\lambda\,] = Energy\,/\,Volume\,.
\end{aligned}$$

So, switching between them involves a derivative,

$$u_\lambda(T,\,\lambda)\cdot d\lambda \;=\; u_f(T,\,f)\cdot df \quad\Longrightarrow\quad u_\lambda(T,\,\lambda) \;=\; u_f(T,\,f)\cdot\frac{df}{d\lambda}$$

$$\text{and} \quad f = \frac{c}{\lambda} \quad\Longrightarrow\quad \frac{df}{d\lambda} = -\,\frac{c}{\lambda^2}\,. \tag{6}$$

So, the energy density over the entire range is found from either integral,

$$u(T) = \int_{f=0}^{\infty} u_f(T,\,f)\;\,df = \int_{\lambda=0}^{\infty} u_\lambda(T,\,\lambda)\;\,d\lambda\,. \tag{7}$$

The $-$ sign in Eqn. (6) compensates for the change in integration direction (as $f\nearrow$, $\lambda\searrow$).

2 Last Time

2.1 Energy density results

Thanks to the theoretical treatment of Wilhelm Wien, we know that the energy density should depend not on T and λ independently, but partly on their product

$$u(T,\, \lambda) \;=\; \frac{1}{\lambda^5}\,\phi(\lambda T)\;. \qquad \left[\;\text{Planck}:\;\phi(\lambda T) = \;\frac{A}{e^{b\,/\,\lambda T}-1}\right]\;. \tag{8}$$

Note: Hal had an error in powers last time, having to do with the $df/d\lambda$ factor I discussed above. This is the correct formula, and he has included it in the slides, so update your notes.

You also showed that the equation of state for the box of thermal radiation (what we now refer to as a *photon gas*) has a **radiation pressure**,

$$P \;=\; \frac{1}{3}\,\frac{U}{V} \;=\; \frac{1}{3}\,u(T)\;, \tag{9}$$

and the energy density summed over all λ satisfies the **Stefan-Boltzmann Law**,

$$u(T) \;=\; \int_{\lambda=0}^{\infty} u(T,\,\lambda)\;d\lambda \;=\; A\cdot T^4\;. \tag{10}$$

2.2 Distribution of energy states

The Boltzmann theory tells us that if a system is in thermal equilibrium with a heat reservoir at temperature T, then the probability of the system having energy E_n is given by

$$P(E_n) \propto e^{-E_n\,/\,kT} \quad\Longrightarrow\quad P(E) \;=\; \frac{e^{-E_n\,/\,kT}}{Z}\;. \tag{11}$$

For now, we treated the denominator Z as a normalization constant. Today, we will derive an expression for it in this system.

Today's main goal is to derive the Planck Radiation formula, namely the expression for $u(T,\,\lambda)$ in to quote Hal, who says it more eloquently than I could,
"The argument is a lovely synthesis of all of the ideas that we have explored in the course: it touches on light as a particle (which we saw in relativity) with $E = pc$; we will use our understanding of light as a wave (which we explored in the waves portion of the course) that causes charged particles to oscillate; we will use all the tools in thermal physics that e have built up over the last several weeks; and finally, this argument opens up an exploration of Quantum Mechanics, which we will spend the last two weeks of the course exploring."

2.3 Finding Z

Suppose I told you, "there are two possible outcomes, and A is twice as likely as B." What are the probabilities of A and of B? Since $P(A) + P(B) = 1$ (total probability),

$$P(B) = \frac{P(B)}{1} = \frac{P(B)}{P(A) + P(B)} = \frac{P(B)}{2 \cdot P(B) + P(B)} = \frac{1}{3} \, .$$
$$\text{So,} \quad P(A) = \tfrac{2}{3} \quad \& \quad P(B) = \tfrac{1}{3} \, .$$

The Boltzmann factor argument leaves us in a similar position, where we know the relative magnitudes of the state probabilities, but not each probability by itself. And we solve it by the same trick as the case above, adding up all the probabilities to 1.

Given a system with a set of possible energies $\{ E_n \}$, each satisfying Eqn. (11), the sum of all probabilities gives us

$$1 = \sum_{all\, n} P(E_n) \; = \; \sum_{all\, n} \left[\frac{e^{-E_n/kT}}{Z} \right] \; = \; \frac{\left[\sum_{all\, n} \, e^{-E_n/kT} \right]}{Z} \, ,$$
$$\text{so} \quad Z = \sum_{all\, n} \, e^{-E_n/kT} \, , \quad \text{and} \quad P(E_n) = \frac{e^{-E_n/kT}}{\sum_{all\, n} \, e^{-E_n/kT}} \tag{12}$$

I could pull Z out of the sum because it is the same constant for each $P(E_n)$ term.

This Z sum is called the **partition function**. In many (most?) cases, you cannot actually perform the sum, but Z is still remarkably useful in Statistical Mechanics. To take one example relevant to today's work, we find the average energy of a system by adding up all the possible energies, weighting each one by its probability of occurring.

$$\bar{E} = \sum_{all\, n} \left[\, E_n \; P(E_n) \,\right] = \; \frac{1}{Z} \cdot \sum_{all\, n} \left[\, E_n \cdot e^{-E_n/kT} \,\right] . \tag{13}$$

So, if we can figure out all of the energies for the photons in our box, then we can find the probabilities and determine the average energy of the system.

3 Possible Energy States

We now have to find a way to identify all of the energy states of the blackbody system. In doing this work around 1900, Max Planck was partly relying on his masterful command of late-19th century theoretical physics (Electromagnetism, Statistical Mechanics, ...), and partly making a bold new assumption that he, frankly, had no justification for. Buckle up.

3.1 Energy states of EM waves

Planck modeled the blackbody as a hollow box filled with EM radiation, in thermal equilibrium with the box walls, which is maintained at temperature T. For simplicity, we will assume that the box is cubic, of side L, but it turns out the shape really does not matter. The walls of the box are perfectly reflective, so any EM wave inside it must have 0-magnitude at the wall. In one dimension, this would mean waves such a those shown in Figure 1.

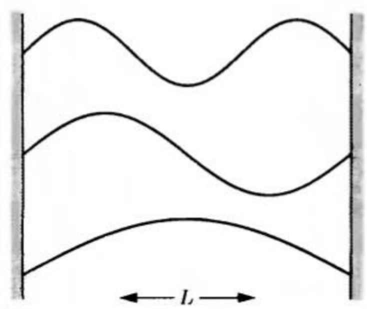

Figure 1: Standing waves in a 1D well.

For a box of length L, the possible standing waves have wavelengths and wave numbers,

$$\lambda_1 = 2\,L,\ \lambda_2 = \frac{2\,L}{2},\ \lambda_3 = \frac{2\,L}{3},\ \ldots\ \ \lambda_n = \frac{2\,L}{n}\ , n = \{1, 2, 3, \ldots\}\ .\ \ k_n = \frac{2\pi}{\lambda_n} = n \cdot \frac{\pi}{L}\ . \tag{14}$$

In 3D, there will be a standing wave in each xyz-direction, so the wave number is a vector,

$$\vec{k}(n_x, n_y, n_z) = n_x\,\frac{\pi}{L}\,\hat{x} + n_y\,\frac{\pi}{L}\,\hat{y} + n_z\,\frac{\pi}{L}\,\hat{z};\,, \qquad |\,\vec{k}\,| = \sqrt{n_x^2 + n_y^2 + n_z^2}\ \ \frac{\pi}{L}\ . \tag{15}$$

$$\lambda(n_x, n_y, n_z) = \frac{2\pi}{|\,\vec{k}\,|} = \frac{2\,L}{\sqrt{n_x^2 + n_y^2 + n_z^2}}\ . \tag{16}$$

We want to find how many states are there in a particular range of wavelengths, $(\lambda, \lambda + d\lambda)$.

To help with this argument, consider this question that came to me today. Suppose you wanted to count how many cars there are on a freight train. You could walk the length of it counting all the way, or you could use the mileage markers by the side of the track to find the total length of the train, ℓ, then divide by the standard car length, $cl = 15\,m/car$. So, if the train were $1200\,m$ long, it would have $\# = \ell\,/\,cl = (1200\,m)\,/\,(15\,m/car) = 80\ cars$.

I am going to make a similar argument here for counting standing wave states. Look at Figure 2, which is a 2D grid of possible (n_x, n_y) combinations, and use your imagination to

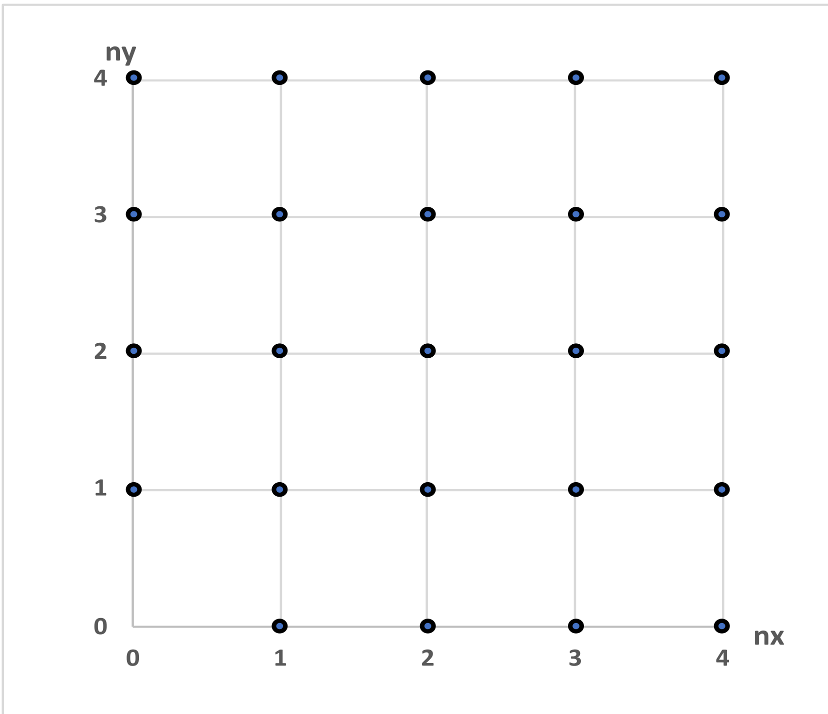

Figure 2: Possible n-combinations in 2D well.

conceptualize a 3D grid of possible (n_x, n_y, n_z) combinations (sorry, no 3D graphics for you). Notice that each point on the $\vec{n}$-grid corresponds to one corner of a unit-box (sides $1 \times 1 \times 1$). So if I were to ask how many points there are in a $|\,\vec{n}\,|$ value within a range $(n, n+dn)$, the answer would just be the "volume" of that shell region in n-space, that is all combinations of (n_x, n_y, n_z) such that

$$\#\text{ states} = \frac{1}{8}\,(\,4\pi\; n^2\,)\; dn\ .$$

Turning that into a question about states in the wavelength range $(\lambda, \lambda + d\lambda)$,

$$|\,\vec{n}\,| = \frac{2\,L}{\lambda} \quad\Longrightarrow\quad |\,dn\,| = \frac{2\,L}{\lambda^2}\; d\lambda\ ,$$
$$\#\text{ states} = \frac{1}{8}\; 4\pi\; \left(\frac{2\,L}{\lambda}\right)^2\; \frac{2\,L}{\lambda^2}\; d\lambda = 4\pi\; L^3\; \frac{d\lambda}{\lambda^4}\ .$$

First, I will point out the $L^3 = V$, the physical volume of the blackbody box. Next, I have to toss in a new piece of information. As you may recall from *Physics 142*, all EM-radiation comes in two independent bf polarization states, say 0° & 90°, or *righthand circular & lefthand circular*. This means that we have to double our answer. The end result is that

$$\text{for wavelengths in the range } (\lambda,\; \lambda + d\lambda)\,, \quad \frac{\#\text{ states}}{V} = \frac{8\pi}{\lambda^4}\; d\lambda\ . \tag{17}$$

This **density of states** calculation was purely within the realm of classical Physics and standard Electromagnetism. Planck's next step was definitely not.

3.2 Planck's Quantization hypothesis

Now Planck turned his attention to the walls of the box. These are emitting and absorbing EM radiation constantly, maintaining the equilibrium distribution in the box. In order to emit light at wavelength λ, the oscillators (atoms?) in the wall would have to vibrate at the corresponding frequency, $f = c/\lambda$. In classical physics, that oscillator could emit any amount energy at that frequency, and if you integrate over all possible frequencies, you get an answer that people already knew gave infinite energy values at short wavelengths, the so-called *Ultraviolet catastrophe*.

Planck made what he thought was a mathematical fix by taking a sum in stead of an integral, He assumed that the oscillator at frequency f could only emit certain amounts of energy,

$$\begin{aligned} E_{emission} &= h\,f \;\text{ or } 2\,h\,f \;\text{ or } 3\,h\,f \;\text{ or } \;\ldots \\ &= \frac{h\,c}{\lambda} \;\text{ or }\; \frac{2\,h\,c}{\lambda} \;\text{ or }\; \frac{3\,h\,c}{\lambda} \;\text{ or }\; \ldots\;. = n\cdot\frac{h\,c}{\lambda} \end{aligned} \tag{18}$$

This constant h was assumed to be small and to have units $[\,h\,] = Energy \cdot Time$ to make the expression an energy. This is a fairly standard physicist trick – approximate a continuous process by a discrete one to make the math easier, then take the limit as $h \to 0$.

But now he had a way to write out the all the possible energies E_n, associated with that oscillator. Now, defining the dimensionless quantity y to simplify our expressions and bringing Boltzmann's ideas to bear (all sums are over $n : 0 \to \infty$), we can find probabilities and the average energy for that oscillator,

$$E_n = n \cdot \frac{h\,c}{\lambda} \quad \Longrightarrow \quad e^{-E_n\,/\,kT} = (\,e^{-hc\,/\,\lambda kT})^n = y^n \ .$$

$$P(E_n) = \frac{e^{-E_n\,/\,kT}}{Z} = \frac{y^n}{\sum\, y^n} \ .$$

$$\bar{E} = \frac{\sum\, E_n\, y^n}{\sum\, y^n} = \frac{h\,c}{\lambda}\,\frac{\sum\, n\, y^n}{\sum\, y^n} \ . \tag{19}$$

But, we figured out those sums in Section 1.1. Using Eqns. (3) & (4),

$$\bar{E} = \frac{h\,c}{\lambda} \cdot \frac{y}{(1-y)^2} \cdot \frac{1-y}{1} = \frac{h\,c}{\lambda} \cdot \frac{y}{1-y} \ ,$$

$$\bar{E} = \frac{h\,c}{\lambda} \cdot \frac{e^{-hc\,/\,\lambda kT}}{1 - e^{-hc\,/\,\lambda kT}} = \frac{h\,c}{\lambda} \cdot \frac{1}{e^{hc\,/\,\lambda kT} - 1} \ . \tag{20}$$

So, we now know the average energy associated with the wavelength λ (Eqn. (20)), and how many states there are with that wavelength (Eqn. (17)). Putting these together gives the complete expression for the energy density in the Planck Blackbody Formula,

$$u(T,\,\lambda) \cdot d\lambda = \frac{8\pi\,h\,c}{\lambda^5} \cdot \frac{1}{e^{hc\,/\,\lambda kT} - 1} \cdot d\lambda \ . \tag{21}$$

Note that the units are correct. The exponential ahas a dimensionless argument, $hc\,/\,\lambda kT$, and the other factors give $(Energy \cdot Length)\,/\,(Length^5) \cdot Length = Energy\,/\,Length^3$.

3.3 Some comments on the Planck formuia

Let me highlight some aspects of the behavior of the Planck Radiation formula.

Small λ limit

For small values of λ, the exponent gets large, which makes the exponential term large,

$$\frac{h\,c}{\lambda\,k\,T} \ll 1 \quad \Longrightarrow \quad e^{hc\,/\,\lambda kT} \gg 1 \ ,$$

$$\frac{8\pi\,h\,c}{\lambda^5} \cdot \frac{1}{e^{hc\,/\,\lambda kT} - 1} \ \longrightarrow \ \frac{8\pi\,h\,c}{\lambda^5} \cdot e^{-hc\,/\,\lambda kT} \ \longrightarrow \ 0 \ ,$$

because the exponential $\to 0$ faster than λ^{-5} grows as $\lambda \to 0$.

Large λ limit
For large values of λ, the exponent gets small, so the exponential term approaches 1,

$$\frac{h\,c}{\lambda\,k\,T} \gg 1 \quad \Longrightarrow \quad e^{hc\,/\,\lambda kT} \approx 1 + \frac{h\,c}{\lambda\,k\,T}\ ,$$
$$\frac{8\pi\,h\,c}{\lambda^5}\cdot\frac{1}{e^{hc\,/\,\lambda kT}-1} \;\longrightarrow\; \frac{8\pi\,h\,c}{\lambda^5}\cdot\frac{\lambda\,k\,T}{h\,c} \;\longrightarrow\; 0\ .$$

Maximum of the distribution
So, $u(T,\,\lambda) \to 0$ is always non-negative, and it approaches 0 at each end of the λ range. It must have an interior maximum. Finding it involves a numerical approximation, but I want to focus on a particular aspect of the solution.

$$\frac{d}{d\,\lambda}\,u(T,\,\lambda) = (8\pi\,h\,c)\cdot\left[\,\frac{-5}{\lambda^6}\cdot\frac{1}{e^{hc\,/\,\lambda kT}-1} - \frac{1}{\lambda^5}\cdot\frac{-h\,c}{\lambda^2\,k\,T}\cdot\frac{e^{hc\,/\,\lambda kT}}{(e^{hc\,/\,\lambda kT}-1)^2}\,\right]$$
$$\text{max:}\;\; 0 = \frac{8\pi\,h\,c}{\lambda^6\cdot(e^{hc\,/\,\lambda kT}-1)^2}\cdot\left[-5\cdot(e^{hc\,/\,\lambda kT}-1) + \frac{h\,c}{\lambda\,k\,T}\cdot e^{hc\,/\,\lambda kT}\right]\ . \tag{22}$$

For this Eqn. 22 to be 0, the square bracket must be zero. But all of those terms depend on the dimensionless variable which I will call w,

$$\text{With}\quad w = \frac{h\,c}{\lambda\,k\,T}, 0 = (w-5)\cdot e^w + 5\ . \tag{23}$$

One lesson is a numerical argument – e^w grows so fast that the only that way Eqn. (23) can be 0 is if w is just a little bit less than 5. The more subtle lesson is that this is saying that

$$\frac{h\,c}{\lambda\,k\,T} \approx 4.95 \quad \Longrightarrow \quad \lambda_{peak} \approx 4.95\cdot\frac{h\,c}{k\,T}\ . \tag{24}$$

The peak has a wavelength value that is inversely proportional to the temperature. This *Wien Displacement Law* was one of the experimental facts that Planck hoped to match.

Integral of the distribution
The integral over λ gives the total energy density at temperature T. Without doing the integral, I want to point out something. I rewrite it in terms of the w defined above,

$$\lambda = \frac{h\,c}{w\,k\,T} \quad \Longrightarrow \quad d\lambda = -\frac{h\,c}{w^2\,k\,T}\ dw \quad (\text{as}\ \ \lambda : 0 \to \infty,\ \ w : \infty \to 0.)$$
$$u(T) = \int_{\lambda=0}^{\infty} u(T,\,\lambda)\ d\lambda = -(8\pi\,h\,c)\int_{w=\infty}^{0}\left(\frac{h\,c}{w\,k\,T}\right)^{-5}\cdot\frac{1}{e^w-1}\,\frac{h\,c}{w^2\,k\,T}\ dw$$
$$u(T) = \left[\frac{8\pi\ k^4}{h^3\ c^3}\right]\cdot T^4\cdot\int_{w=0}^{\infty}\frac{w^3}{e^w-1}\,dw\ . \tag{25}$$

Eqn. (25) divides $u(T)$ into three distinct terms, each with their own significance.

- The first term gives a numerical relationship in terms of three of the primary physical constants of nature: k, h, & c.
- The second term gives the primary meaningful relationship between the two physical quantities, $u(T) \propto T^4$. This captures the *Stefan-Boltzmann Law*, another experimental fact that Planck's theory needed to match.
- The third term is a dimensionless integral, *i.e.*, it is a pure number. As it happens, its value is $\pi^4/15$, but that is almost inconsequential. The important lesson for us is that we have isolated the crucial physics in the first two terms, and the rest is "just math".

Reference: *Modern Physics*, by Randy Harris, Second Edition (1998). Appendix C.