Homework 1

Due Thursday, September 3rd at 5pm

Read Chapter 5 of Taylor's Classical Mechanics.

1. A massless spring has unstretched length ℓ_o and force constant k. One end is attached to the ceiling and a mass m is hung from the other. The equilibrium length of the spring is now ℓ_1. (**a**) Write down the condition that determines ℓ_1. Suppose now the spring is stretched a further distance x beyond its new equilibrium length. Show that the net force (spring plus gravity) on the mass is $F = -kx$. That is, the net force obeys Hooke's law, when x is the distance from the equilibrium position—a very useful result, which lets us treat a mass on a vertical spring just as if it were horizontal. (**b**) Prove the same result by showing that the net potential energy (spring plus gravity) has the form $U(x) = \text{const} + \frac{1}{2}kx^2$.

2. (**a**) Write down the potential energy $U(\phi)$ of a simple pendulum (mass m, length ℓ) in terms of the angle ϕ between the pendulum and the vertical. (Choose the zero of U at the bottom.) Show that, for small angles, U has the Hooke's law form $U(\phi) = \frac{1}{2}k\phi^2$, in terms of the coordinate ϕ. What is k?

   (**b**) A second, unusual pendulum is made by fixing a string to a horizontal cylinder of radius R, wrapping the string several times around the cylinder, and then tying a mass m to the loose end. In equilibrium the mass hangs a distance ℓ_o vertically below the edge of the cylinder. Find the potential energy if the pendulum has swung to an angle ϕ from the vertical. Show that for small angles, it can be written in the Hooke's law form $U = \frac{1}{2}k\phi^2$. Comment on the value of k.

3. Consider a simple harmonic oscillator with period τ. Let $\langle f \rangle$ denote the average value of any variable $f(t)$, averaged over one complete cycle:

   $$\langle f \rangle = \frac{1}{\tau}\int_0^\tau f(t)dt. \tag{1}$$

   Prove that $\langle T \rangle = \langle U \rangle = \frac{1}{2}E$ where E is the total energy of the oscillator. [Hint: Start by proving the more general, and extremely useful, results that $\langle \sin^2(\omega t - \delta) \rangle = \langle \cos^2(\omega t - \delta) \rangle = \frac{1}{2}$. Explain why these two results are almost obvious, then prove them by using trig identities to rewrite $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos(2\theta)$.]

4. The potential energy of a particle moving in one dimension and with mass m at a distance r from the origin is

   $$U(r) = U_o\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right) \tag{2}$$

   for $0 < r < \infty$, with U_o, R, and λ all positive constants. Find the equilibrium position r_o. Let x be the distance from equilibrium and show that, for small x, the PE has the form $U = \text{const} + \frac{1}{2}kx^2$. What is the angular frequency of small oscillations?