Mechanics
Day 11

Outline

I Noether Table

II Conservation of Energy I

III Conservation of Energy II

Didn't get to this. Derive E directly from Noether's result

IV. What about the cat?

I Noether Table

Symmetry		Cons. Law
Translation in time	$\longleftrightarrow$	Energy
Translation in space	$\longleftrightarrow$	Momentum
Rotations	$\longleftrightarrow$	Ang. Mom.
Gauge trans. (Global)	$\longleftrightarrow$	?
Gauge trans. (local)	$\longleftrightarrow$	?

II A General Lagrangian

$$\mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$

changes in time both because of $q_i(t)$, $\dot{q}_i(t)$ and the t-dependence:

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \quad (1)$$

Meanwhile the E-L eqn.s give

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{d}{dt} p_i = \dot{p}_i$$

Putting this into (1) gives

$$\frac{d\mathcal{L}}{dt} = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \frac{d}{dt}\left(\sum_i p_i \dot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t}$$

Now, what would it take for $\mathcal{L}$ to be time translation invariant?

This is precisely the condition

$$\frac{\partial \mathcal{L}}{\partial t} = 0.$$

Then

$$\frac{d}{dt}\left(\sum_i p_i \dot{q}_i - \mathcal{L}\right) = 0$$

or

$$\mathcal{H} \equiv \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L} = \text{const.}$$

We call $\mathcal{H}$ the Hamiltonian and in many cases it is equal to the total energy.

So,

$$\boxed{\text{If } \mathcal{L} \text{ does not depend explicitly on time, } \frac{\partial \mathcal{L}}{\partial t}=0, \text{ then } \mathcal{H} \text{ is conserved.}}$$

When does $\mathcal{H} = E$?

Suppose

$$\vec{r}_\alpha = \vec{r}_\alpha(q_1, \ldots, q_n) \leftarrow \text{no } t \text{ dependence}$$

sometimes called "natural",

then

$$\mathcal{H} = T + U$$

Pf: $T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,2}$

Compute

$$\dot{\vec{r}}_\alpha = \sum_{i=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_i}\dot{q}_i$$

$$= \sum_j A_{ij}\dot{q}_j$$

and

$$\dot{\vec{r}}_\alpha^{\,2} = \sum_j \left(\frac{\partial \vec{r}_\alpha}{\partial q_j}\dot{q}_j\right) \cdot \sum_k \left(\frac{\partial \vec{r}_\alpha}{\partial q_k}\dot{q}_k\right)$$

Then

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,2} \equiv \frac{1}{2}\sum_{j,k} A_{jk}\dot{q}_j \dot{q}_k$$

with

$$A_{jk} \equiv A_{jk}(q_1, \ldots, q_n) = \sum_\alpha m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_j}\right)\cdot\left(\frac{\partial \vec{r}_\alpha}{\partial q_k}\right)$$

This is the only term with $\dot{q}$ dependence so,

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2}\sum_{j,k}\left(A_{jk}\delta_{ij}\dot{q}_k + A_{jk}\dot{q}_j\delta_{ik}\right)$$

But, then

$$\sum_i p_i \dot{q}_i = \sum_i \left(\sum_j A_{ij}\dot{q}_j\right)\dot{q}_i$$

$$= \sum_{i,j} A_{ij}\dot{q}_i \dot{q}_j = 2T$$

Finally,

$$\mathcal{H} = 2T - (T - U)$$

$$= T + U = E. \checkmark$$

We discussed two aspects of this derivation in detail. The first was the meaning of the dot product

$$\dot{\vec{r}}_\alpha^{\,2} = \sum_j \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j \right) \cdot \sum_k \left(\frac{\partial \vec{r}_\alpha}{\partial q_k} \dot{q}_k \right).$$

Recall $\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$, so in more detail this dot product is

$$\dot{\vec{r}}_\alpha^{\,2} = \sum_{j,k} \left(\frac{\partial x_\alpha}{\partial q_j}, \frac{\partial y_\alpha}{\partial q_j}, \frac{\partial z_\alpha}{\partial q_j} \right) \cdot \left(\frac{\partial x_\alpha}{\partial q_k}, \frac{\partial y_\alpha}{\partial q_k}, \frac{\partial z_\alpha}{\partial q_k} \right) \dot{q}_j \dot{q}_k .$$

We also discussed the distinction between a free index and a dummy index. Consider the indexed equation

$$v^i = \sum_{j=1}^{N} M^i_j u^j$$

We call i a "free" index and j a "dummy" index. The name dummy refers to the fact that the symbol or name for this index can always be changed. This is because whatever its name, the sum still indicates that the index takes every value from 1 to N. So we can write

$$v^i = \sum_{k=1}^{N} M^i_k u^k .$$

The rule for free indices is that they have to match on both sides of an equation. So we can also rename i, say $i \to l$, but when we do we have to do it on both sides:

$$v^l = \sum_{j=1}^{N} M^l_j u^j .$$

These distinctions are invaluable for interpreting and manipulating indexed equations. These techniques were used to arrive at $p_i = \sum_j A_{ij} \dot{q}_j$.

IV You were correct in guessing that the way the cat turns over is by changing its shape. Next time, we will discuss this in detail. In the mean time we note that this is closely related to gauge theories. A gauge theory is a description of a physical system with certain mathematical redundancies that have no physical meaning.

The exactly vanishing conserved quantity indicates that the symmetry in question represents a redundancy in the mathematical description with no physical import.

While gauge = redundancy it is not useless. Often we can couple two systems using a gauge variable and they turn out to be efficient for many situations and are often useful in the quantization procedure.

The last entries of our Noether table are:

Symmetry		Cons. law
Gauge Trans. (Global)	↔	Charge conservation
Gauge Trans. (Local)	↔	0 ← zero

after the coupling the variable can acquire physical meaning.

These ideas become more and more useful as you study advanced physics.

We will explore these ideas in some detail for the example of the cat turning over.