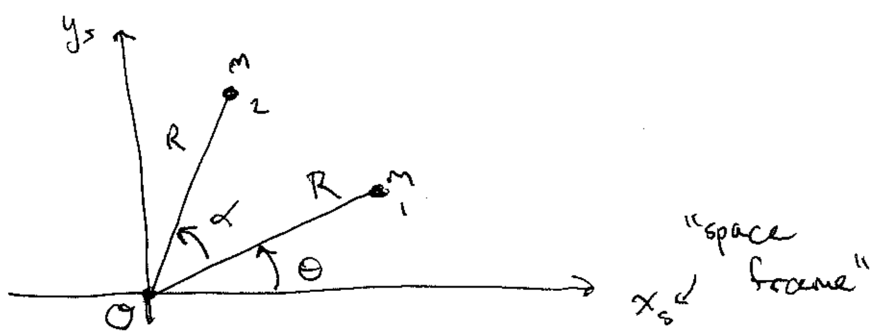
Mechanics Day 12

Outline

I Pictures of the cat

II A simple mechanical analog

III A better Analog



The rods are massless and pinned at O so that they can only rotate

α = shape coordinate

θ = Orientational coordinate

A muscle, motor or some other agent can change α, the shape, by acting at O but never generates any external torque. Simplest possible model of a "flexible" body.

I As suggested by our class discussions the cat turns over by changing its shape — see photos from Frohlich (1980), Sci. Amer.

II Let's study an analogous mechanical system made up of two rods of length R and with two masses m at their ends.

We take 1 and 2 to be distinguishable and so α and $2\pi - \alpha$ are distinct configurations.

So, $\alpha \in [0, 2\pi]$ or $\alpha \in S^1$

With these assumptions we choose

$$L_{sz} = 0$$

in analogy with the cat. Then

$$L_{sz} = m\,(x_{s1}\dot{y}_{s1} - y_{s1}\dot{x}_{s1}) + m\,(x_{s2}\dot{y}_{s2} - y_{s2}\dot{x}_{s2})$$

Or in our angular coord.s

$$x_{s1} = R\cos\theta\,, \quad x_{s2} = R\cos(\theta+\alpha)$$

$$y_{s1} = R\sin\theta\,, \quad y_{s2} = R\sin(\theta+\alpha)$$

and $L_{sz} = mR\left(c^2\theta\,\dot{\theta} + s^2\theta\,\dot{\theta}\right) + mR\left(c^2(\theta+\alpha)(\dot{\theta}+\dot{\alpha}) + s^2(\theta+\alpha)(\dot{\theta}+\dot{\alpha})\right)$

Upon $L_{sz} = 0$, we get,

$$\dot{\theta} + \dot{\theta} + \dot{\alpha} = 0$$

$$\Rightarrow \quad \dot{\theta} = -\tfrac{1}{2}\dot{\alpha}$$

We see that a change of shape ($\dot{\alpha}$) requires a change of orientation ($\dot{\theta}$) in order to maintain $L_{sz} = 0$.

The ang. mom. of 1 during a change must cancel that of 2 and hence the bisector of α is fixed.

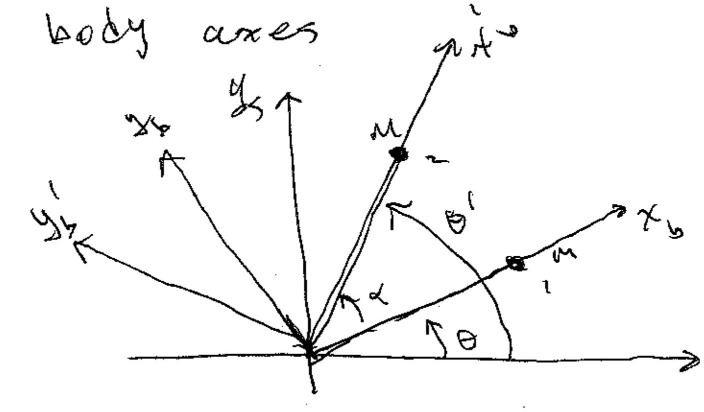
Then if 2 moves forward (ccw) by $\tfrac{1}{2}\Delta\alpha$ 1 moves back by $\tfrac{1}{2}\Delta\alpha$ (cw) and θ changes by $\tfrac{1}{2}\Delta\alpha$.

θ is just a function of α and hence there's a 1-to-1 correspondence.

This model is too simple to describe the cat. When we return to the initial shape α_0 after a deviation the constancy of the bisector implies that the final orientation (θ_0) is the same as the initial one θ_0.

Mathematically: $\theta = -\tfrac{1}{2}\alpha + \text{const}$,

One more point: Intro body axes



Clearly,

$$\theta' = \theta + \alpha \qquad \text{(trans)}$$

Attaching a body frame to a flexible body is conventional, we call it a <u>gauge convention</u>. And (trans) is called a gauge transformation.

α is gauge invariant

θ is not.

Now we have

$$x_{s3} = Rc(\theta+\alpha) + Rc(\theta+\alpha+\beta)$$

$$y_{s3} = Rs(\theta+\alpha) + Rs(\theta+\alpha+\beta)$$

and so after a fair bit of algebra

$$L_{sz} = 0 = (4+2c\beta)\dot{\theta} + (3+2c\beta)\dot{\alpha} + (1+c\beta)\dot{\beta}.$$

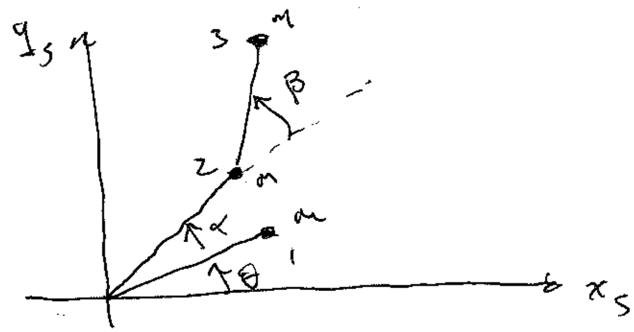
If we define

$$\dot{\theta} = A_\alpha \dot{\alpha} + A_\beta \dot{\beta}$$

then we find

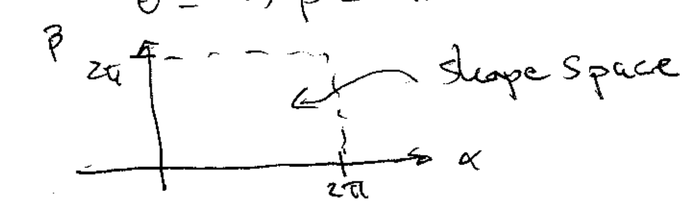
$$A_\alpha = -\frac{3+2c\beta}{4+2c\beta} \quad \text{and} \quad A_\beta = -\frac{1+c\beta}{4+2c\beta}.$$

III Let's consider



Shape space is now 2D.

$$0 \le \alpha, \beta \le 2\pi$$

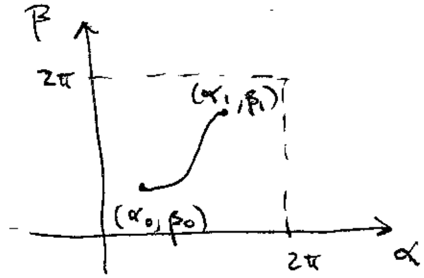


We see that in this case the amount that θ changes depends on where you are in shape space.

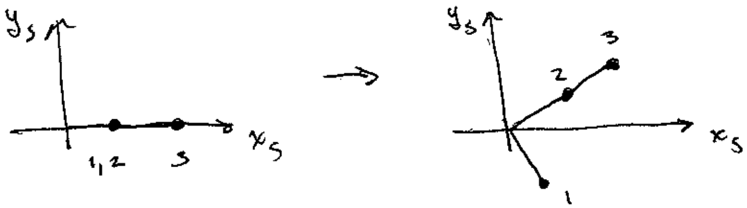
To calculate the total change in θ, call it $\Delta\theta$, we multiply through by dt and integrate

$$\Delta\theta = \int A_\alpha \, d\alpha + \int A_\beta \, d\beta.$$

Notice that if we choose a path of configurations in shape space, such as

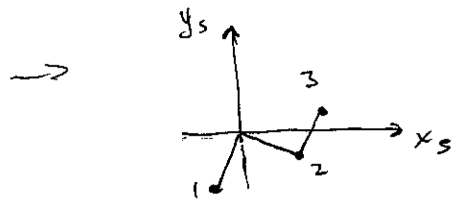


the result of calculating $\Delta\theta$ is independent of how fast you traverse the path. For this reason $\Delta\theta$ is called a "geometric phase", or a Berry phase in quantum mechanics.



From (i) → (ii)

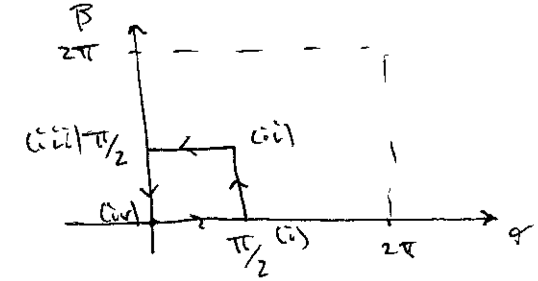
$$\Delta\theta = \int_0^{\pi/2} -\frac{1+c\beta}{4+2c\beta}\, d\beta = -\frac{\pi}{4} + \frac{\pi}{6\sqrt{3}} = -27.7^\circ$$



From (ii) → (iii)

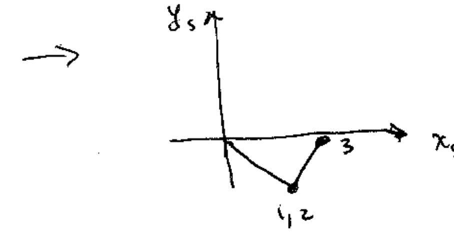
$$\Delta\theta = \int_{\pi/2}^{0} -\frac{3+2c\beta}{4+2c\beta}\, d\alpha \Big|_{\beta=\pi/2} = \frac{3\pi}{8} = 67.5^\circ$$

Let's see if we can use this analog to reproduce the cat trick. Consider the following closed path in shape space



From (0) → (i) we have

$$\Delta\theta = \int_0^{\pi/2} A_\alpha\, d\alpha = \int_0^{\pi/2} -\frac{3+2c\beta}{4+2c\beta}\, d\alpha \Big|_{\beta=0} = -75^\circ$$



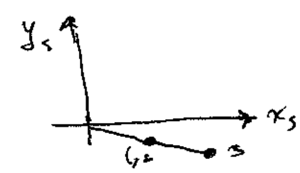
Finally, from (iii) → (iv) and

$$\Delta\theta = \int_{\pi/2}^{0} -\frac{1+c\beta}{4+2c\beta}\, d\beta = 27.7^\circ$$

Adding them all up we get

$$\Delta\theta_{tot} = -7.5^\circ$$

and we get



the cat trick!