Mechanics
Day 13

<u>Outline</u>

I Where are we?
Where are we going?

II Central Forces: Reduction

In fact, we've only seen half of the power of the last point:

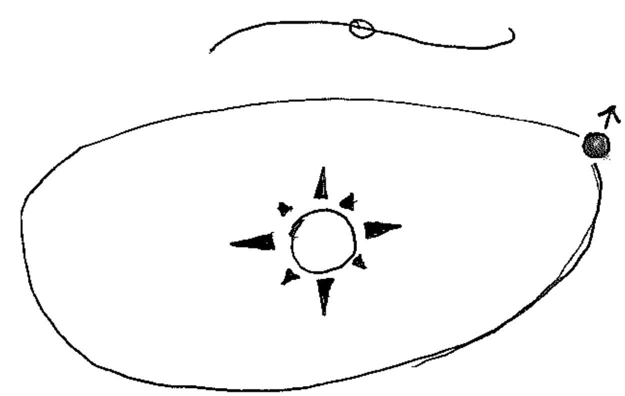
symmetry $\Longleftrightarrow$ Cons. Law

The Second facet of this advantage is the ability to remove D.O.F. altogether. We call this process reduction. We turn now to one of the most important examples

I We've now touched on all three of the great advantages of Lagrangian Mechanics:

- Can use generalized coord.s
- Completely elliminate the forces of constraint.
- Marvelous dual role of symmetry.

in all of physics, but also for illustrating the power of reduction.



$$\delta S = 0$$

is just another way of saying that S is stationary for the physical path (similar to $df = 0$ in calculus). But we know what the conditions for

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2, t)\, dt$$

to be stationary are:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad (i = 1, 2)\ !$$

<u>The constraints have been completely</u>

II <u>Central Forces: Reduction</u>

We've setup the general formalism, now, let's apply it! Major application: the two body problem. — two bodies that experience an interaction force, between one another, but no external forces.

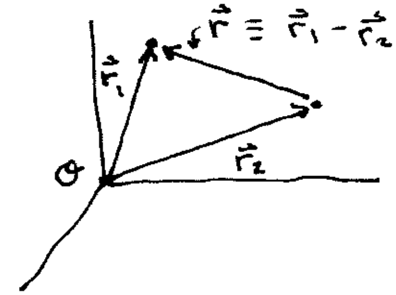
<u>Examples</u>: $U = -\dfrac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$ gravitation

or

$U = \dfrac{k\, q\, Q}{|\vec{r}_1 - \vec{r}_2|}$, $k = \dfrac{1}{4\pi\epsilon_0}$ Coulomb force

We've done it. Lagrangian mechanics provides a completely independent formulation of mechanics that is equivalent to Newton's. From here forward we can <u>use whatever formulation is most convenient</u>. I will do this!

3D space



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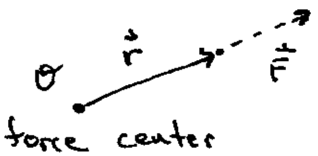
Our present (remarkable) goal is <u>Reduction</u>: show that these problems of 6 D.O.F. are equivalent to a 1 D.O.F. problem. This is not due to constraints; instead we will use symmetry and conservation laws (Noether's ideas).

<u>Useful Definition:</u> A central force is one for which the force is always directed toward or away from a fixed "force center." If we take the center as the origin, $\mathcal{O}$, then these are ~~(inward and/or outward)~~ radial forces:

$$\vec{F}(\vec{r}) = f(\vec{r})\,\hat{r}$$ ← magnitude; can be + or − and depends on $\vec{r}$

Pictorially



If we assume our central force is derivable from a potential, $\vec{F} = -\vec{\nabla} U$, we have an interesting consequence for $f(\vec{r})$. In spherical coordinates

$$\vec{F} = -\vec{\nabla} U = -\frac{\partial U}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

but $\vec{F}$ is central, so,

$$\frac{\partial U}{\partial \theta} = 0 \text{ and } \frac{\partial U}{\partial \phi} = 0 \Rightarrow U = U(r) = U(|\vec{r}|)$$ ← U is spher. symmetric!

therefore

$$f(\vec{r}) \equiv -\frac{\partial U(r)}{\partial r} = f(|\vec{r}|) = f(r)$$

and

$$\vec{F} = f(r)\,\hat{r} \quad \left(\begin{array}{c}\text{conservative} \\ \text{central} \\ \text{force}\end{array}\right)$$

We'll often drop 'conservative' but this isn't a great practice. <u>End of useful definition.</u>

In both examples above U only depends on $|\vec{r}_1 - \vec{r}_2| \equiv |\vec{r}| = r$ and we will call $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ the "relative position." This dependence is a manifestation of the translational invariance of these problems; Mathematically,

$$\vec{r}_1 \to \vec{r}_1 + \vec{\epsilon} \quad \text{and} \quad \vec{r}_2 \to \vec{r}_2 + \vec{\epsilon}$$

gives rise to exactly the same potential and hence force btwn the two particles.

Noether's thm. ⇒ there is a corresponding conserved quantity. Here we can't translate the particles independently but <u>only</u> both together and this corresponds to conservation of <u>total momentum</u>. (check it using Noether's thm!)

This all indicates that we want conglomerate general coordinates in addition to our relative ones $\vec{r}$.

Introduce the CM (center of mass) position:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

Then

check it

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} + \frac{m_2\vec{r}_1 - m_2\vec{r}_2}{M}$$

$$= \frac{(m_1+m_2)\vec{r}_1}{M} = \vec{r}_1 \checkmark$$

and

$$\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

Put this all into the (conservative) central force Lagrangian,

$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - U(r)$$

$$= \frac{1}{2}M\dot{\vec{R}}^2 + \left(\frac{1}{2}\frac{m_1 m_2}{M}\dot{\vec{r}}^2 - U(r)\right)$$

$$\equiv \mathcal{L}_{cm} + \mathcal{L}_{rel}$$

The quantity $\frac{m_1 m_2}{M}$ has units of mass and a convenient short hand is

$$\mu \equiv \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}$$

We call μ the "reduced mass." Then

$$\mathcal{L}_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

is a Lagrangian that only depends on $\vec{r}$ and $\dot{\vec{r}}$.

Let's look at E.O.M.s. For $\vec{R}$ we have

$$\frac{\partial\mathcal{L}}{\partial R_i} = 0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{R}_i}\right) = M\ddot{R}_i \quad (i=1,2,3)$$

or

$$M\ddot{\vec{R}} = 0.$$

This implies $M\dot{\vec{R}} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2$

$= \vec{p}_1 + \vec{p}_2 = \vec{P}_{tot}$ is constant!

The center of mass moves as if it were a free particle (at a constant velocity ~~speed~~).

Idea: change reference frame to one in which $\dot{\vec{R}} = 0$ and $\vec{R} = 0$ (CM at origin) then

$$\mathcal{L} = \mathcal{L}_{rel}$$

Now, $\mathcal{L}$ only depends on $\vec{r}$ $\Rightarrow$ we're down to 3 D.O.F.

There is still a rotational symm: $|\vec{R}(\vec{r}_1-\vec{r}_2)| = |(\vec{r}_1-\vec{r}_2)|$
Then, Noether's theorem implies that the total angular momentum is conserved:

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

In the CM frame $\dot{\vec{R}} = 0$ and $\vec{R} = 0$, so,

$$\vec{r}_1 = \frac{m_2}{M}\vec{r} \quad \text{and} \quad \vec{r}_2 = -\frac{m_1}{M}\vec{r}$$

and

$$\vec{L} = \frac{m_2}{M}\vec{r} \times \frac{m_1 m_2}{M}\dot{\vec{r}} + \left(-\frac{m_1}{M}\right)\vec{r} \times m_2\left(-\frac{m_1}{M}\right)\dot{\vec{r}}$$

$$= \frac{m_1 m_2}{M^2}\left(m_2 \vec{r}\times\dot{\vec{r}} + m_1 \vec{r}\times\dot{\vec{r}}\right) = \vec{r}\times\mu\dot{\vec{r}}$$

By convention we choose the z-axis to be parallel to $\vec{L}$. Then all of the motion lies in the xy-plane (viz. $\vec{r}$ and $\dot{\vec{r}}$ lie in plane perp. to $\vec{L}$).

We have reduced the problem to one in 2 D.O.F. at this point.

Now choose polar coordinates in the xy-plane (r, ϕ). Our Lagrangian is,

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

The <u>vector</u> $\vec{L}$ is constant, which means both its magnitude and its direction are constant.
Because $\vec{L} = \vec{r}\times\mu\dot{\vec{r}}$ it must be perpendicular to both $\vec{r}$ and $\dot{\vec{r}}$. As long as $\vec{L} \neq 0$, the two vectors $\vec{r}$ and $\dot{\vec{r}}$ span a plane.

<u>Convention</u>: We are free to orient our axes w/in the CM frame however we like.

We notice that ϕ is ignorable

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{const.}$$

$$\Rightarrow \quad \mu r^2 \dot{\phi} = \text{const} \equiv l \quad \text{(Eq. 1)}$$

This is just the z-component of angular momentum that we knew we had around (recall $\mu r^2 = I$ for pt. particle and $\dot{\phi} = \omega$ = angular speed, $L_z = I\omega$). Initial conditions set the value of l and so we

Can use (Eq. 1) to get rid of ϕ,

$$(\text{Eq. } 1) \Rightarrow \dot{\phi} = \frac{l}{\mu r^2},$$

once we're working with the E.O.M.

This will complete our reduction,

all that remains is 1 D.O.F., r.

Reduction has been a major success for central forces; we've achieved the best possible outcome: 6 D.O.F. $\rightarrow$ 1 D.O.F.