Mechanics Day 14

Outline

0. Announcements: Tutoring shift: Th 2-4, W 7-9pm

I Last time

II Radial E.O.M. for the Kepler Problem

III Qualitative Analysis

IV Solving the radial E.O.M

I. Intro'd coord.s

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \left(\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}\right)$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \qquad \left(\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}\right)$$

for relative and CM positions.

- Transformed to CM frame: $\dot{\vec{R}} = 0$ and choose $\vec{R} = 0$ origin.
- Found

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\frac{m_1 m_2}{M}\dot{r}^2 - U(r) + \frac{1}{2}\mu r^2\dot{\phi}^2$$

where $\mu = \frac{m_1 m_2}{M}$ is the reduced mass.

Regarding the last term

$$\frac{1}{2}\mu r^2\dot{\phi}^2 \rightsquigarrow \mu r^2\dot{\phi} = \ell \quad (\text{Eq. } 1)$$

I made a small infelicity last time.

You do not want to replace

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

in the Lagrangian, which could lead to confusing results, like $\frac{\partial \mathcal{L}}{\partial \phi} = 0$. Instead you

can use (Eq. 1) to get rid of $\dot\phi$,

$$(\text{Eq. 1}) \Rightarrow \dot\phi = \frac{l}{\mu r^2},$$

once we're working with the E.O.M.

This will complete our reduction, all that will remain is 1 D.O.F., r. Reduction has been a major success for central forces; we've achieved the best possible outcome

$$6 \text{ D.O.F.} \rightarrow 1 \text{ D.O.F.}$$

$\underline{\text{II}}$ Radial E.O.M. for the Kepler problem.

We have,

$$U = -\frac{G m_1 m_2}{r}$$

and so,

$$\mathcal{L} = \frac{1}{2}\mu \dot r^2 + \frac{1}{2}\mu r^2 \dot\phi^2 + \frac{G m_1 m_2}{r}.$$

Then,

$$\frac{\partial \mathcal{L}}{\partial r} = \mu r \dot\phi^2 - \frac{G m_1 m_2}{r^2}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) = \frac{d}{dt}(\mu \dot r) = \mu \ddot r.$$

Now we proceed to solving the 1 E.O.M for r. There is one <u>caveat</u>; we're no longer working with the physically intuitive variables $\vec r_1$ and $\vec r_2$. This means that we have to decode what we learn along the way.

Let's specialize to the Kepler problem (see your book for general case).

So,

$$\mu \ddot r = \mu r \dot\phi^2 - \frac{G m_1 m_2}{r^2}$$

Now let's use l from above,

$$\mu \ddot r = \mu r \cdot \frac{l^2}{\mu^2 r^4} - \frac{G m_1 m_2}{r^2}$$

$$= \frac{l^2}{\mu r^3} - \frac{G m_1 m_2}{r^2}$$

In this form it's tempting to think of the first term on the R.H.S. as another radial force, the "centrifugal force",

$$F_{cf} \equiv \frac{l^2}{\mu r^3}$$

Because this term actually originated with the kinetic energy we call it a "fictitious" force (note it's just $\mu r\dot{\phi}^2 = \frac{\mu v_\phi^2}{r}$, with $v_\phi = r\dot{\phi}$, from 7A).

We can also derive this force from a potential,

$$U_{cf} = \frac{\ell^2}{2\mu r^2},$$

will do the trick: $F_{cf} = -\frac{d}{dr} U_{cf}$.

III Qualitative Analysis

There's one more conservation law hiding in the woodwork. What is it? What's the remaining symmetry? Answers: The symmetry is time translation invariance and it corresponds to conservation of energy (note that $\mathcal{L}$ doesn't explicitly depend on t):

$$E = K.E. + P.E.$$

$$= \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\phi}^2 + U(r)$$

$$= \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

$$= \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}(r).$$

We'll use this conservation law to extract as much qualitative information about the motion as possible.

Thus

$$\mu\ddot{r} = -\frac{d}{dr}\left(U_{cf} + U\right)$$

and we call the combination

$$U_{cf} + U \equiv U_{eff}$$

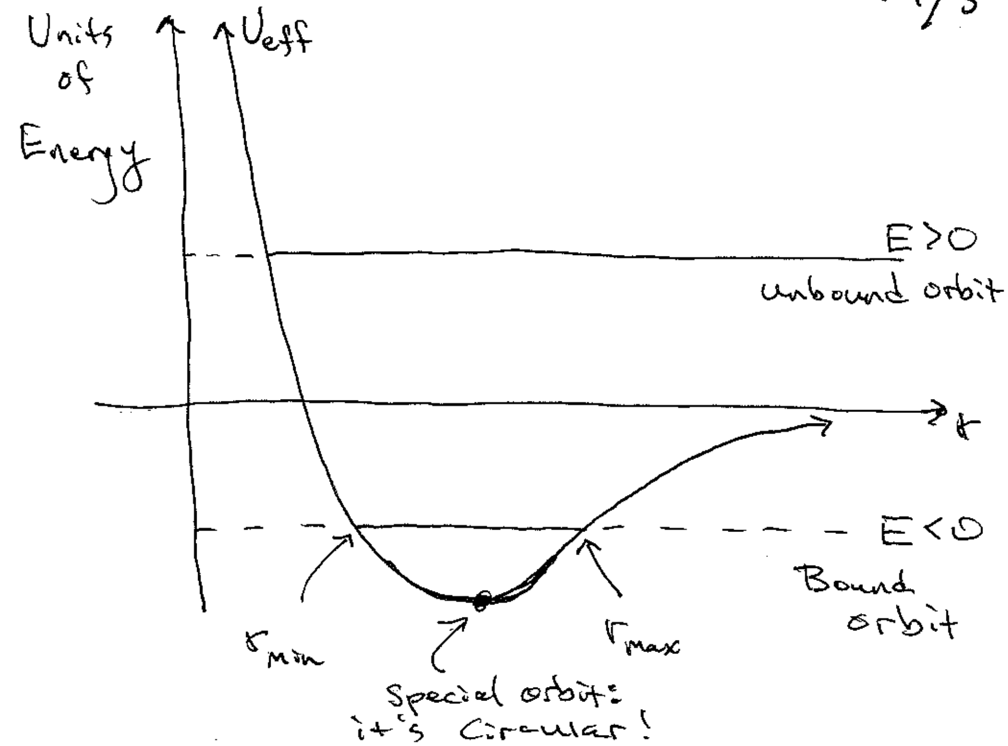
the "effective potential". The introduction of U_{cf} is, of course, useful for any central force problem.

Then the effective potential for the Kepler problem is

$$V_{eff} = -\frac{G m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

dominates at large r

dominates at small r



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IV Solving the radial E.O.M.

At this point we need to invoke two pieces of mathematical cleverness to solve our E.O.M.

The first one is somewhat intuitive, we are going to change the parametrization from $r(t)$ to $r(\phi)$ — this will tell us the geometric shape of the orbit rather than giving us ~~a~~ its time parametrization.

In practice this means we need to use the chain rule to replace d/dt with $d/d\phi$,

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot\phi \frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi}.$$

The second bit of cleverness is not obvious, we'll see why it's nice in the end.

We introduce a new variable,

$$u \equiv 1/r \Rightarrow r = 1/u.$$

Recall
Our radial eq. ~~is then~~

$$\mu \ddot r = \frac{\ell^2}{\mu r^3} - \frac{G m_1 m_2}{r^2}$$

We need:

$$\dot r = \frac{\ell}{\mu r^2}\frac{dr}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right) = \frac{\ell u^2}{\mu}\left(-\frac{1}{u^2}\right)\frac{du}{d\phi}$$

$$= -\frac{\ell}{\mu}\frac{du}{d\phi}$$

Also,

$$\ddot r = \frac{d}{dt}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell^2}{\mu^2 r^2}\frac{d^2u}{d\phi^2}$$

$$= -\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\phi^2}$$

Putting this into the E.O.M. gives

$$-\mu\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\phi^2} = \frac{\ell^2 u^3}{\mu} - G m_1 m_2 u^2$$

$$\Rightarrow \frac{d^2u}{d\phi^2} = -u + \frac{G m_1 m_2 \mu}{\ell^2}$$

Driven oscillator, use $u_{gen} = u_p + u_h \Rightarrow u(\phi) = A\cos(\phi-\delta) + \frac{G m_1 m_2 \mu}{\ell^2}$