Mechanics

Day 15

Outline

I Last time

II A little notation

III Bounded Kepler Orbits

IV Relation between energy and eccentricity

V Unbound orbits

VI Orbital Transfer

I. • Found the radial E.O.M. for Kepler problem

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{G m_1 m_2}{r^2}$$

• Qualitative analysis led to

$$V_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{G m_1 m_2}{r}$$

• Used $r(\phi)$ and $u = 1/r$ to solve it:

$$u'' = -u + \frac{\gamma\mu}{\ell^2}$$

with $\gamma \equiv G m_1 m_2$, $\mu = \frac{m_1 m_2}{M}$, $\ell = \mu r^2 \dot{\phi}$

and

$$u(\phi) = A\cos(\phi - \delta) + \frac{\gamma\mu}{\ell^2}$$

By choosing origin of ϕ we elliminate δ,

$$u(\phi) = A\cos(\phi) + \frac{\gamma\mu}{\ell^2}$$

Let $c \equiv \frac{1}{K} = \frac{\ell^2}{\gamma\mu}$, $[c] = \text{length}$

and, $\epsilon \equiv \frac{A\ell^2}{\gamma\mu} = A \cdot c$ unitless "eccentricity"

Then,

$$u(\phi) = \frac{1}{r(\phi)} = \frac{\epsilon}{c}\cos(\phi) + \frac{1}{c}$$

$$= \frac{1}{c}(\epsilon\cos\phi + 1)$$

or

$$\boxed{r(\phi) = \frac{c}{1 + \epsilon\cos\phi}}$$

This equation we have studied! It's an harmonic oscillator with a (constant) forcing term. Introduce two shorthands

$$u' \equiv \frac{du}{d\phi} \quad \text{and let} \quad \gamma \equiv G m_1 m_2$$

then
$$u'' = -u + \frac{\gamma\mu}{l^2}.$$

General solution to an inhomogeneous equation = gen. sol. of homog. + part. sol.

II

A bit more notation:

let $c \equiv \frac{1}{K} = \frac{l^2}{\gamma\mu}$ $\quad [c] = $ length

and $\epsilon \equiv \frac{A l^2}{\gamma\mu} = A \cdot c$ $\quad$ Unitless "eccentricity"

$$\Rightarrow u(\phi) = \frac{1}{r(\phi)} = \frac{\epsilon}{c}\cos(\phi) + \frac{1}{c}$$
$$= \frac{1}{c}\left(1 + \epsilon\cos\phi\right)$$

or

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}$$

Guess a particular solution

$$u_p = \text{const.} \overset{\text{call it}}{=} K$$

Then $u_p' = 0$, $u_p'' = 0$ and

$$0 = -K + \frac{\gamma\mu}{l^2} \Rightarrow K = \frac{\gamma\mu}{l^2}$$

The general solution is,

$$u(\phi) = A\cos(\phi - \delta) + \frac{\gamma\mu}{l^2}$$

and by choosing the origin of ϕ properly we can get rid of δ,

$$u(\phi) = A\cos(\phi) + \frac{\gamma\mu}{l^2}$$

III Bounded Kepler orbits

Note that if $\epsilon < 1$ then r is bounded for all ϕ but if $\epsilon \geq 1$ then at some ϕ r runs off to infinity. We will find that this is the difference btwn the bound and unbound orbits. Let's focus on bound orbits first, $\epsilon < 1$. The bounds on r are at $\phi = 0$ and $\phi = \pi$

We have,

$\phi = 0$

$$r_{min} = \frac{c}{1+\epsilon}$$

"perihelion"

$\phi = \pi$

$$r_{max} = \frac{c}{1-\epsilon}$$

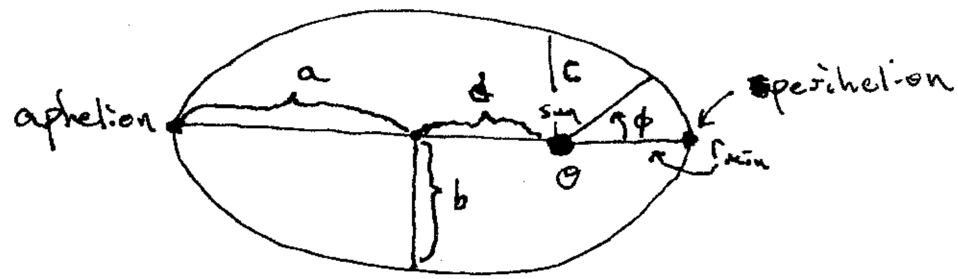
"aphelion"

On the homework you will show

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \left(\begin{array}{c}\text{Equation of an}\\ \text{ellipse}\end{array}\right)$$

for this orbit. The various constants are

$$a = \frac{c}{1-\epsilon^2} \qquad b = \frac{c}{\sqrt{1-\epsilon^2}} \qquad d = a\epsilon$$



$a + d = r_{max}$. Note that our origin is at one of the foci, this is the origin of the "$x+d$" above.

Indeed,

$$\frac{b}{a} = \frac{c}{\sqrt{1-\epsilon^2}} \; \frac{1-\epsilon^2}{c} = \sqrt{1-\epsilon^2}$$

and this verifies that ϵ is the eccentricity. Note the limits

$\epsilon \to 0 \Rightarrow b/a = 1 \Rightarrow$ a circle

and $\epsilon \to 1 \Rightarrow b/a \to 0$ a highly stretched ellipse. From this it follows that $d = a\epsilon$ is indeed the distance to a focus.

IV. How does this geometry relate to the physics?

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2}$$

$$= \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right)$$

But

$$r_{min} = \frac{c}{1+\epsilon} = \frac{\ell^2}{\gamma\mu(1+\epsilon)}$$

Putting this into the equation for E we have,

$$E = \frac{\gamma\mu(1+\epsilon)}{2\ell^2}\left(\gamma(1+\epsilon) - 2\gamma\right)$$

$$= \frac{\gamma\mu}{2\ell^2}(1+\epsilon)(\gamma\epsilon - \gamma)$$

$$= \frac{\gamma^2\mu}{2\ell^3}(\epsilon^2 - 1)$$

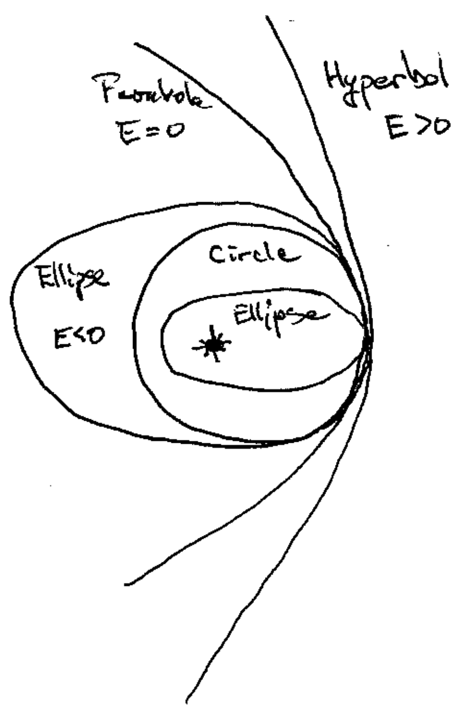
$$= \frac{\gamma}{2c}(\epsilon^2 - 1)$$

Note that this prefactor is always positive

So indeed for $\epsilon < 1$ $E < 0$ and the orbit is bounded. While for $\epsilon > 1$ $E > 0$ and the orbit is unbounded.

V Now, assume $\epsilon \geq 1$ then ϕ_{max} is determined by

$$1 + \epsilon\cos\phi_{max} = 0 \Rightarrow \epsilon\cos\phi_{max} = -1$$



and $r(\phi) \to \infty$ as $\phi \to \pm\phi_{max}$. These orbits are generally hyperbolae (with a parabolic orbit when $\epsilon = 1$). The demonstration of these claims, namely that

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

is very similar to your HW problem (Taylor 8.16) and I've spared you the demonstration but check it if you're up for it.

Summary

Orbit Eq: $r(\phi) = \dfrac{c}{1+\epsilon\cos\phi}$

w/ $c = \dfrac{l^2}{\gamma\mu}$ $\quad \gamma = G m_1 m_2$

$$\mu = \frac{m_1 m_2}{(m_1+m_2)}$$

Energy/eccentricity: $E = \dfrac{\gamma^2\mu}{2l^2}(\epsilon^2-1) = \gamma/2c\,(\epsilon^2-1)$

Eccen.	Energy	Orbit
$\epsilon = 0$	$E = -\frac{\gamma^2\mu}{2l^2} < 0$	circle
$0<\epsilon<1$	$E<0$	ellipse
$\epsilon = 1$	$E = 0$	parabola
$\epsilon > 1$	$E > 0$	hyperbola

III Orbital Transfer

A common problem for satellite engineers is the transfer of a satellite from one orbit to another.

The same analysis as before yields

$$r(\phi) = \frac{c}{1+\epsilon\cos(\phi-\delta)}$$

although closest approach to earth is called perigee and the most distant pt of the orbit is apogee.

We retain δ now because we can't, in general, align the x-axis with both perigees in an orbit transfer.

The idea is that the satellite rockets give a brief strong impulse to the satellite that changes its orbit. Call this a thrust.

We assume we know the change in velocity due to this thrust from which we can find $E_1 \to E_2$ and $l_1 \to l_2$.