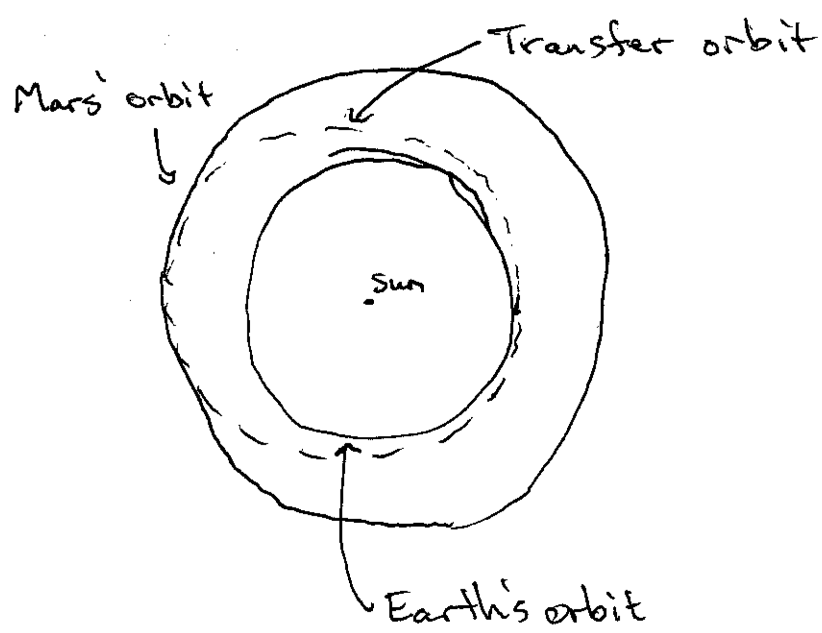
Mechanics
Day 17

Outline

I Last time

II Noninertial Frames



I • Orbital Transfer

- Tangential thrust

$$\epsilon_2 = \lambda^2 \epsilon_1 + (\lambda^2 - 1)$$

- Hohmann transfer

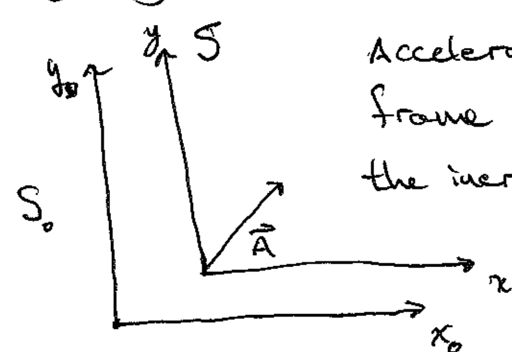
$$\lambda = \sqrt{\frac{2R_m}{R_e + R_m}}$$

- All flows from orbit equation

$$r(\phi) = \frac{c}{1 + \epsilon\cos(\phi - \delta)}$$

II • Newton's laws only hold in inertial frames

• We are careful to write down the Lagrangian in an inertial frame



Acceleration of frame S w.r.t. the inertial frame S

For the next few meetings we explore non-inertial frames

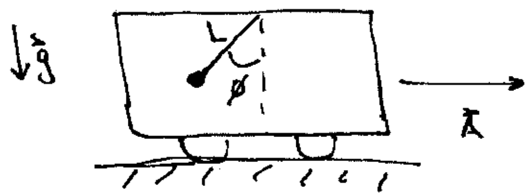
Consider the position of a particle. In the frame S_0 this is measured to be $\vec{r}_0$ and it obeys Newton's 2nd law

$$m\ddot{\vec{r}}_0 = \vec{F}$$

In S they measure $\vec{r}$ and

$$\dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V}$$

particle's vel. w.r.t. ground = particles vel. w.r.t. moving frame + vel. of moving frame w.r.t. ground



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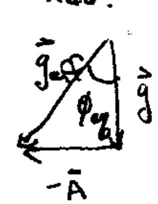


Noninertial (boxcar) frame:

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\vec{A}$$
$$= \vec{T} + m(\vec{g}-\vec{A})$$
$$= \vec{T} + m\vec{g}_{eff}\,, \quad \vec{g}_{eff} \equiv \vec{g}-\vec{A}$$

Then

$$\phi_{eq} = -\arctan\left(\frac{A}{g}\right)$$

and

$$\omega = \sqrt{\frac{g_{eff}}{L}} = \sqrt{\frac{\sqrt{g^2+A^2}}{L}}$$

This then implies,

$$\ddot{\vec{r}}_0 = \ddot{\vec{r}} + \vec{A} \quad \text{or} \quad \ddot{\vec{r}} = \ddot{\vec{r}}_0 - \vec{A}$$

so that

$$m\ddot{\vec{r}} = m\ddot{\vec{r}}_0 - m\vec{A}$$
$$= \vec{F} - m\vec{A}$$

This looks a lot like Newton's 2nd law. In fact it is if we agree to introduce an "inertial force"

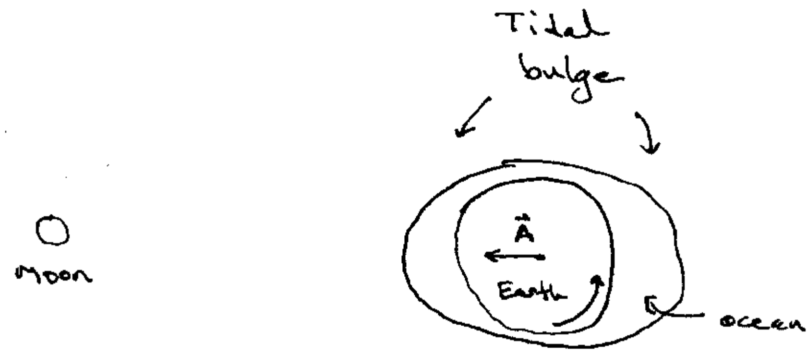
$$\vec{F}_{inertial} = -m\vec{A}$$

Examples: Airplane take off, elevator, car, etc.

This is computationally fast, conceptually subtle route!

III Great Example: Tides

The Earth is a noninertial frame because it is rotating. We will discuss this more next time. However, it is also a noninertial frame because it is accelerating towards the moon (sun also). The tides are a combined effect of the moon's gravitational attraction of the ocean and of this acceleration.



The acceleration $\vec{A}$'s magnitude and direction are determined by treating the Earth and moon as point masses concentrated at their respective centers.

Forces on m: (1) Earth's gravity: $m\vec{g}$
(2) Moon's gravity: $-GM_m m\, \hat{d}/d^2$ (M_m = moon's mass)
(3) Net non-gravitational force: $\vec{F}_{ng}$

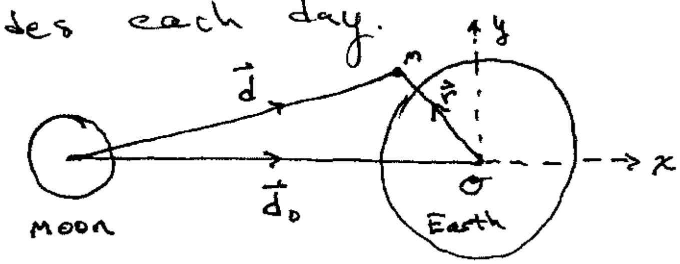
And the earth is a noninertial frame with acceleration,

$$\vec{A} = -\frac{GM_m}{d_0^2}\hat{d}_0$$

We can use our noninertial frame law

$$m\ddot{\vec{r}} = \vec{F} - m\vec{A}$$

On the other hand the gravitational pull of the moon on the oceans is <u>greater</u> on the side closer to the moon and <u>weaker</u> on the opposite side. This is why there are two high tides each day.

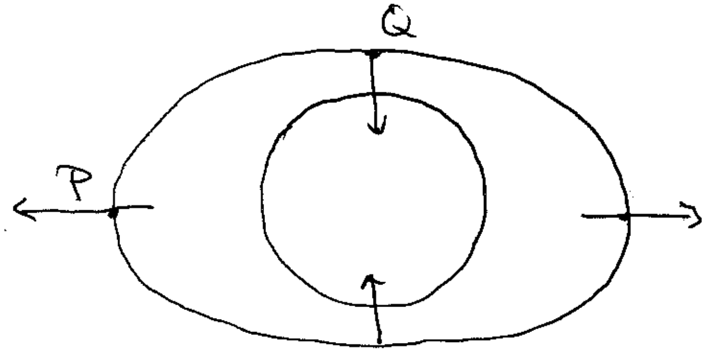


$$\Rightarrow m\ddot{\vec{r}} = \left(m\vec{g} - GM_m m\frac{\hat{d}}{d^2} + \vec{F}_{ng}\right) + GM_m m\frac{\hat{d}_0}{d_0^2}$$

$$\Rightarrow m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{ng} \underbrace{- GM_m m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right)}_{\equiv \vec{F}_{tid}}$$

Illustrate direction of tidal force on figure — See next page.

The direction of $\vec{F}_{tid}$ is interplay of $\hat{d}_0$ and $\hat{d}$, d^2 and d_0^2:



while if $f = f(x,y,z)$ then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \vec{\nabla} f \cdot d\vec{r}$$

Now, consider a level set

$$f(x,y,z) = c \quad (\text{a const.})$$

If we choose a displacement within this surface then $d\vec{r}$ is tangent

To find the height of the tides we'll leverage a nice argument: the tidal bulge is an equipotential surface.

Mathematical prerequisite:

Level set of f: $\quad f = \text{const.}$

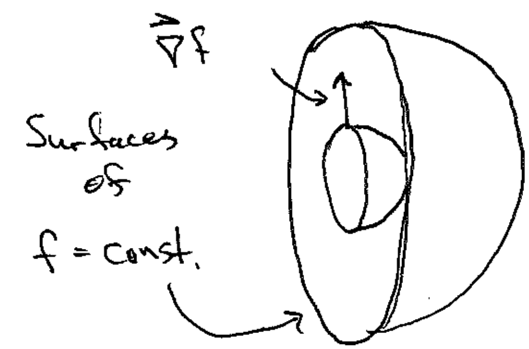
If $g = g(x)$ then

$$dg = \frac{dg}{dx}dx$$

to $f = c$ and, certainly $df = 0$.

Then,

$$df = 0 = \vec{\nabla} f \cdot d\vec{r}$$



This means that $\vec{\nabla} f$ is normal (i.e. perpendicular) to $f = c$. Example:

$$f(x,y,z) = x^2+y^2+z^2, \quad \vec{\nabla} f = 2(x,y,z) \propto \hat{r} \; \checkmark.$$

An equipotential is just a level set of the potential energy function. We're going to argue that $\vec{\nabla} U$ is normal to the tidal bulge and hence that the tidal bulge is an equipotential. (Note; reversal of mathematical argument just given.)

Write in terms of potentials:

$$m\vec{g} = -\vec{\nabla} U_{eg} \qquad \vec{F}_{tid} = -\vec{\nabla} U_{tid}$$

potential of Earth's grav. pull

$$U_{tid} = -G M_M m \left(\frac{1}{d} + \frac{x}{d_o^2} \right)$$

d_o is const.

and

$$U = U_{eg} + U_{tid}$$

Then $m\vec{g} + \vec{F}_{tid} = -\vec{\nabla} U$ is

Consider a water droplet, it is subject to (in our Earth frame): $m\vec{g}$, $\vec{F}_{tid}$, and $\vec{F}_p$, the pressure of the surrounding water. Water can't exert a shear force $\Rightarrow \vec{F}_p$ is normal to surface of water. But our droplet is in equilibrium and so

$$m\vec{g} + \vec{F}_{tid}$$

is also normal to the water surface.

normal to the surface of the water and indeed it must be an equipotential. This means that

$$U(Q) = U(P)$$

$$\Rightarrow U_{tid}(Q) - U_{tid}(P) = U_{eg}(P) - U_{eg}(Q)$$

$$= mgh$$

where h is the height of the tidal bulge. What about $U_{tid}(Q)$ and $U_{tid}(P)$?

For Q: $d = \sqrt{d_o^2 + r^2}$ and $r \approx R_e$, the radius of the Earth. So

$$U_{tid}(Q) = -GM_m m\left(\frac{1}{\sqrt{d_o^2+R_e^2}} + 0\right)$$

$$= \frac{-GM_m m}{d_o}\left(1+R_e^2/d_o^2\right)^{-1/2}$$

$$\approx -\frac{GM_m m}{d_o}\left(1-\frac{R_e^2}{2d_o^2}\right)$$

For P: $d \approx d_o - R_e \quad x \approx -R_e$

and

$$U_{tid}(P) = -GM_m m\left(\frac{1}{d_o - R_e} - \frac{R_e}{d_o^2}\right)$$

$$= -\frac{GM_m m}{d_o}\left(\frac{1}{1-R_e/d_o} - \frac{R_e}{d_o}\right)$$

$$\approx -\frac{GM_m m}{d_o}\left(1+\frac{R_e}{d_o}+\frac{R_e^2}{d_o^2}-\frac{R_e}{d_o}\right)$$

$$= -\frac{GM_m m}{d_o}\left(1+\frac{R_e^2}{d_o^2}\right)$$

Putting it together

$$-\frac{GM_m m}{d_o}\left(1-\frac{R_e^2}{2d_o^2}\right) + \frac{GM_m m}{d_o}\left(1+\frac{R_e^2}{d_o^2}\right)$$

$$= \frac{3GM_m m R_e^2}{2d_o^3} = mgh$$

Now, $g = GM_e/R_e^2$ and so

$$\frac{3GM_m \cancel{m} R_e^2}{2d_o^3} = \frac{GM_e}{R_e^2}h$$

$$\Rightarrow h = \frac{3M_m R_e^4}{2M_e d_o^3}$$

$\Rightarrow h = 54$ cm [moon alone]

Sun also contributes, exact same analysis with sun's mass and distance yields

$h = 25$ cm [sun alone].