Mechanics

Day 18

Today's Outline:

I Background on Rotational Motion

II Rotating Frames

III Example of centrifugal force: Newton's bucket

For the last two lectures we've focused on noninertial frames that are linearly accelerated. This week we will concentrate on noninertial rotating frames. Once again we will find extra "inertial forces" that modify Newton's 2nd law in the rotating frame:

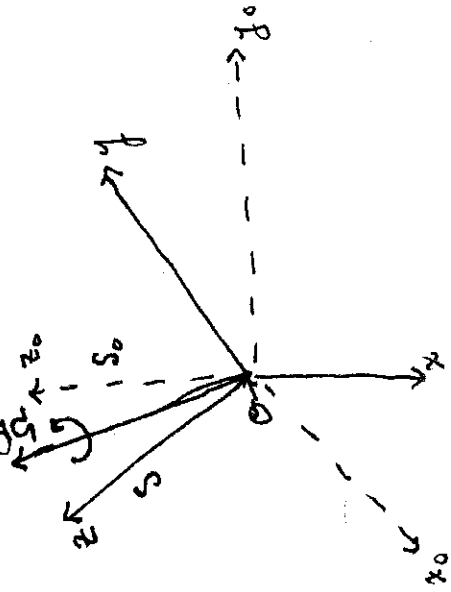$$m\ddot{\vec{r}} = \vec{F} + \vec{F}_{cor} + \vec{F}_{cf}$$

where

$\vec{F}_{cor}$ is the Coriolis force

and $\vec{F}_{cf}$ is the centrifugal force

I Background on Rotational Motion

• Rigid body = N particles whose collective shape doesn't change.

• The most general motion of any body s.t. a point of that body remains fixed, say pt. O, is a rotation about some axis through O; called Euler's rotation theorem.

• Recall definition of $\vec{\omega}$: magnitude ω is the angular speed (e.g. $d\phi/dt$) and direction is given by right hand rule.
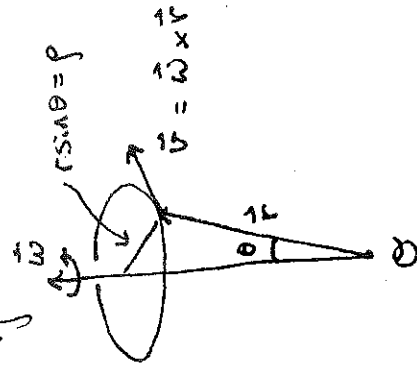
• Useful vector identity:

$v = r\dot{\phi} = r\sin\theta\,\omega$

and points tangentially to circle

$$\Rightarrow \vec{v} = \vec{\omega} \times \vec{r}$$

$r\sin\theta = \rho$

$\vec{v} = \vec{\omega} \times \vec{r}$

More generally, for any vector $\hat{e}$ fixed in the body we have

$$\frac{d\hat{e}}{dt} = \vec{\omega} \times \hat{e}$$

Notation: Following your book, I use

$\vec{\omega}$ for angular velocity of a body

$\vec{\Omega}$ for angular velocity of noninertial, rotating frame

(Recall $\vec{v}, \vec{a}$ and $\vec{v}, \vec{a}$.)

Consider any vector $\vec{Q}$ (could be a position, a force, a momentum etc), we would like to relate these two quantities:

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\begin{array}{l}\text{rate of change of } \vec{Q} \\ \text{relative to } S_0\end{array}\right)$$

and

$$\left(\frac{d\vec{Q}}{dt}\right)_{S} = \left(\begin{array}{l}\text{rate of change of } \vec{Q} \\ \text{relative to } S\end{array}\right)$$

To do this we will use

$$\vec{Q} = Q_1\hat{e}_1 + Q_2\hat{e}_2 + Q_3\hat{e}_3 = \sum_{i=1}^{3} Q_i\hat{e}_i$$

II Rotating Frames:

S_0 : inertial frame

S : rotating frame, ang. vel. $\vec{\Omega}$

Time derivatives:

where $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ are three unit orthogonal vectors that are constant in the rotating frame, i.e. a basis for S.

This expansion still holds in S_0 but there the $\hat{e}_i$ are time dependent.

Now, we can calculate

$$\left(\frac{d\vec{Q}}{dt}\right)_s = \sum_i \frac{dQ_i}{dt}\hat{e}_i$$

There is no decoration necessary here because the Q_i are the same in either frame.

Newton's 2nd law in a rotating frame:

In S_0: ~~$\boxed{M\ddot{\vec{r}} = \vec{F}}$~~ or

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F}$$

Need to work out second derivatives.

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}$$

Then,
$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left(\frac{d\vec{r}}{dt}\right)_{S_0}$$

Then,
$$\vec{F} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0}$$
$$= m\ddot{\vec{r}} + 2m\vec{\Omega}\times\dot{\vec{r}} + m\dot{\vec{\Omega}}\times\vec{r} + m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

or
$$m\ddot{\vec{r}} = \vec{F} - 2m\vec{\Omega}\times\dot{\vec{r}} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$
$$= \vec{F} + 2m\dot{\vec{r}}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$
$$\equiv \vec{F} + \vec{F}_{cor} + \vec{F}_{cf}$$

$$\vec{F}_{cor} = 2m\dot{\vec{r}}\times\vec{\Omega} \qquad \vec{F}_{cf} = m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\vec{e}_i + \sum_i Q_i\left(\frac{d\vec{e}_i}{dt}\right)_{S_0}$$

$$= \sum_i \frac{dQ_i}{dt}\vec{e}_i + \sum_i Q_i(\vec{\Omega}\times\vec{e}_i)$$

$$= \sum_i \frac{dQ_i}{dt}\vec{e}_i + \vec{\Omega}\times\vec{Q}$$

$$= \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega}\times\vec{Q}$$

This is very useful for calculations in rotating frames.

$$\Rightarrow \left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

$$= \left(\frac{d}{dt}\right)_S\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right] + \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_S + 2\vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Simplify notation, reserve "." dot notation,

$$\left(\frac{d\vec{Q}}{dt}\right)_S \equiv \dot{\vec{Q}}$$

dot notation for "our frame" that is the rotating frame S.

III Example of centrifugal force:

The centrifugal force only depends on position not on motion. Its consequences are often geometrical. Your book discusses how the centrifugal force changes the free fall acceleration near earth at length. Here's another example:

Spin a bucket of water about its vertical axis with angular speed Ω. Find the shape of the water's surface once it has reached equilibrium.
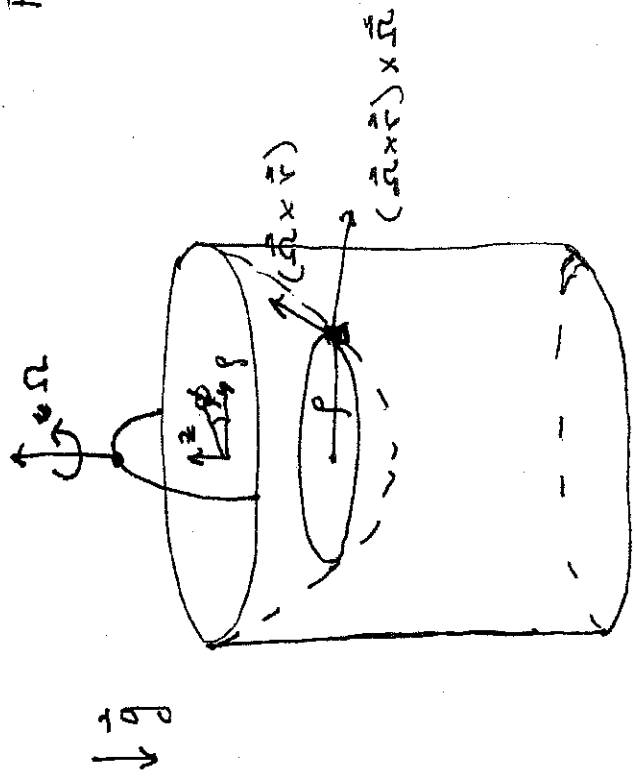

In the water's frame it is at rest and so $\dot{\vec{r}} = 0$ and $\vec{F}_{cor} = 0$. Then

$$m\ddot{\vec{r}} = \vec{F} + \vec{F}_{cf}$$
$$= m\vec{g} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

Applying this to a little parcel of water near the water's surface we find

$$m\ddot{\vec{r}} = -mg\hat{z} + m\Omega^2\rho\,\hat{\rho}$$

this surface is a equipotential. So let's express it in terms of a potential

$$\therefore \ddot{\vec{r}} = -\vec{\nabla}\left(mgz + \tfrac{1}{2}m\Omega^2\rho^2\right) = -\vec{\nabla}U$$

Then the equation of the surface is

$$U = \text{const.}$$

$$\Rightarrow mgz = \text{const.} + \tfrac{1}{2}m\Omega^2\rho^2$$

$$\boxed{\Rightarrow (z - z_0) = \tfrac{1}{2}\frac{\Omega^2}{g}\rho^2}$$

A parabola!

Just as in our discussion of tides