Mechanics. Day 19,

Today's Outline:

I Last Lecture

II The Earth's frame

a) Centrifugal force

b) Coriolis force

[Last time I promised to do Newton's bucket and didn't get to it; instead I'll assign it as homework.]

We treat these fictitious forces just as we would any other forces. The only tricky thing to get the hang of is their directions. To master this we'll treat the Earth's frame in detail.

I. Last Lecture

We've found how Newton's 2^{nd} law is modified for two diffent non-inertial frames

$$m\ddot{\vec{r}} = \vec{F} - m\vec{A} \quad \left(\begin{matrix}\text{acceleration} \\ \text{w/out rotation}\end{matrix}\right)$$

and

$$m\ddot{\vec{r}} = \vec{F} + \vec{F}_{cor} + \vec{F}_{cf} \quad \left(\begin{matrix}\text{rotating} \\ \text{frame}\end{matrix}\right)$$

where

$$\vec{F}_{cor} = 2m\dot{\vec{r}} \times \vec{\Omega}$$

$$\vec{F}_{cf} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

II Earth's frame: Centrifugal force

$$\text{Earth's } \Omega = \frac{2\pi \text{ rad}}{24 \times 3600\text{ s}} \approx 7.3 \times 10^{-5} \text{ rad/s}$$

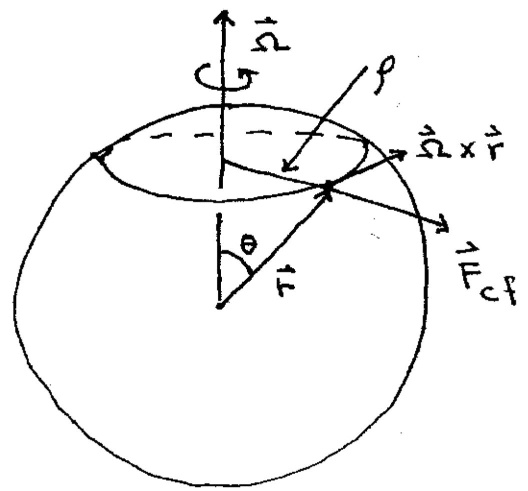
For slow moving ($v \ll 1000$ mi/hr) objects the coriolis force is negligible and we can focus on just $\vec{F}_{cf}$.

To work out the direction of $\vec{F}_{cf} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$ see Figure 1:

$$\vec{F}_{cf} = m\Omega^2 r\sin\theta\,\hat{\rho} = m\Omega^2 \rho\,\hat{\rho},$$

where ρ is the cylindrical radial coordinate.

Figure 1



$\vec{F}_{cf} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$.

$\vec{\Omega} \times \vec{r} = \Omega r \sin\theta \hat{\phi}$ $\quad (\vec{\Omega} \times \vec{r}) \times \vec{\Omega} = \Omega^2 r \sin\theta \sin\frac{\pi}{2} \hat{\rho}$

$= \Omega^2 r \sin\theta \hat{\rho}$

with M and R the mass and radius of the Earth and $\vec{g}_0$ the acceleration just due to gravity $\vec{g}_0 = GM/R^2(-\hat{r})$.

Put in $\vec{F}_{cf}$ to get, $\quad |\vec{r}| \approx R$

$$\vec{F}_{eff} = \vec{F}_{grav} + \vec{F}_{cf} = m\vec{g}_0 + m\Omega^2 R \sin\theta \hat{\rho}$$

thus

$$\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta \hat{\rho}$$

At the equator $\vec{g}$ is smaller by about 3% (check it).

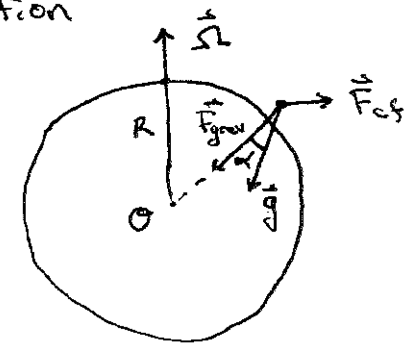
<u>Subtle example</u>: Free-fall

$\vec{g} \equiv$ initial acceleration, relative to earth, of object released from rest in vacuum near Earth's surface.

<u>relative to earth</u> $\Rightarrow$

$$m\ddot{\vec{r}} = \vec{F}_{grav} + \vec{F}_{cf}$$
$$= -\frac{GMm}{R^2}\hat{r} + \vec{F}_{cf}$$
$$= m\vec{g}_0 + \vec{F}_{cf}$$

Cross section



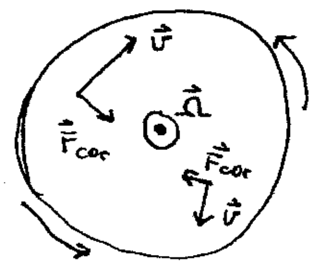
α is wildly exagerated and depends on colatitude θ.

α_{max} is at $\theta = 45°$ and

$\alpha_{max} = 0.0017\,\text{rad} \approx 0.1°$

(check it)

<u>Coriolis force</u>: In this case the direction is easier, there's just one cross product.



$\vec{\Omega}$ out of page

Simple turn table with pucks sliding on it.

Mention Falkland Island story.

<u>Free-fall again</u>: Now,

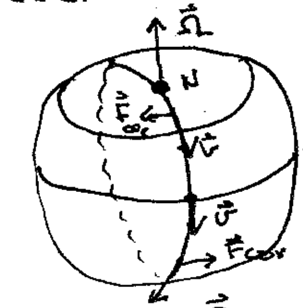
$$m\ddot{\vec{r}} = m\vec{g}_0 + \vec{F}_{cf} + \vec{F}_{cor}$$

$$= m\vec{g} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

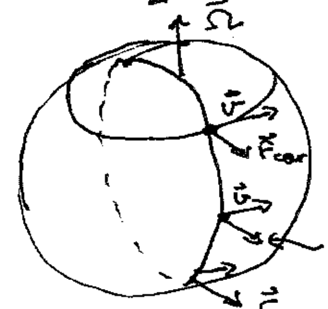
$$\Rightarrow \ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}}\times\vec{\Omega}$$

This equation only depends on $\ddot{\vec{r}}$ and $\dot{\vec{r}}$ $\Rightarrow$ we can arbitrarily shift our origin.

This can be a little subtle too, however:

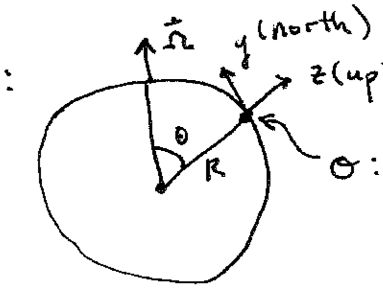


on equator with this $\vec{v}$ no $\vec{F}_{cor}$



In this case $\vec{F}_{cor}$ is always in $\hat{\rho}$ direction <u>but</u> that locally projects differently.

Cross-section:



O: put origin here.

Then $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$ and

$\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta)$, so that

$$\dot{\vec{r}}\times\vec{\Omega} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & \Omega s\theta & \Omega c\theta \end{vmatrix} = (\dot{y}\Omega c\theta - \dot{z}\Omega s\theta, -\dot{x}\Omega c\theta, \dot{x}\Omega s\theta)$$

The E.O.M. is then

$$\ddot{x} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$$
$$\ddot{y} = -2\Omega\dot{x}\cos\theta$$
$$\ddot{z} = -g + 2\Omega\dot{x}\sin\theta$$

which is a set of <u>coupled</u> <u>differential</u> <u>equations</u>. Not easy to solve, so we will make a series of approximations:

First $\Omega \ll 1$, so let's take $\Omega \approx 0$

$$\Rightarrow \ddot{x} = 0 \quad \ddot{y} = 0 \quad \text{and} \quad \ddot{z} = -g$$

with solution,

$$x = 0 \quad y = 0 \quad z = h - \tfrac{1}{2}gt^2$$

assuming $z(0) = h$, $\dot{z}(0) = 0$. This is called the ~~get next~~ zeroth order approximation because it only has $(\Omega)^0 = 1$ in it. To get the first order we put the zeroth order R.H.S of the back into the E.O.M. to find

$$\ddot{x} = 2\Omega g t\sin\theta, \quad \ddot{y} = 0, \quad \ddot{z} = -g$$

$$\Rightarrow \dot{x}(t) = \Omega g t^2\sin\theta + \cancel{\dot{x}_0}^{0} \Rightarrow x(t) = \tfrac{1}{3}\Omega g t^3\sin\theta + \cancel{x_0}^{0}$$

First order is then

$$x(t) = \tfrac{1}{3}\Omega g t^3\sin\theta \quad y = 0 \quad z = h - \tfrac{1}{2}gt^2$$

This reasonably has only $\Omega^0 = 1$ and $\Omega^1 = \Omega$ in it. We could continue in this manner to get as many powers of Ω as we wanted. How big is this effect?

Drop a pebble down a 100 meter mine shaft at the equator and we have $z = h - \tfrac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}}$

while

$$x = \tfrac{1}{3}\Omega g\left(\frac{2h}{g}\right)^{3/2} \approx 2.2\,\text{cm},$$

generally a small effect.

This example illustrates:

- how to calculate cross-products (a reminder)

and

- One approach to solving coupled differential equations.