Sep 2^{nd}, 2015

#1/5

Classical Mechanics

Day 2

Outline:

I Last time

II Standard guess

III Damped Oscillations

IV Regimes

I Oscillations around stable equilibria are ubiquitous

$$K = U''(x_{eq})$$

and we began to investigate the standard guess

$$x(t) = e^{rt}$$

II S.H.O.

$$m\ddot{x} = -Kx$$

$$\dot{x} = re^{rt}, \quad \ddot{x} = r^2 e^{rt}$$

So,

$$mr^2 \cancel{e^{rt}} = -K\cancel{e^{rt}}$$

$$\Rightarrow \quad r = \pm i\sqrt{\frac{K}{m}} \equiv \pm i\omega_0$$

General solution is a linear combination (superposition):

Generally true – 2^{nd} order ODE has general soln depending on 2 constants. Physically it means that you need boundary conditions:

$x(0) = C_1 + C_2$

$\dot{x}(0) = i\omega_0 C_1 - i\omega_0 C_2$

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

constants

Euler's exquisite creation:

$$e^{i\pi} + 1 = 0$$

Generally,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Amazing secret

$$e^{i\theta} = c\theta + is\theta$$

Can use this to derive all of trigonometry!

<u>Example</u>: $e^{i2\theta} = c2\theta + i s2\theta$

$= (e^{i\theta})^2 = (c\theta + i s\theta)^2$

$= c^2\theta - s^2\theta + 2i\, c\theta \cdot s\theta$

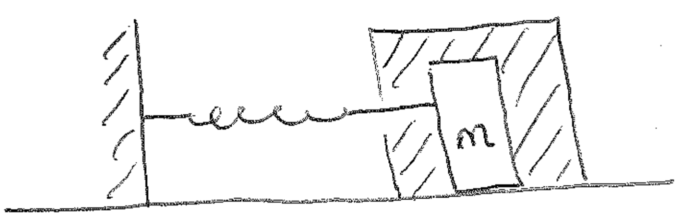
Real parts equal implies

$c2\theta = c^2\theta - s^2\theta = 2c^2\theta - 1$ etc.

III Damped Harmonic Oscillations

Hooke's law: $F_{spring} = -kx$

Immerse the mass in a tank of viscous medium.



<u>Challenge</u>: For the duration of this course never look up a trig. formula

<u>Challenge 2</u>: Figure out four distinct ways to write the general solution to

$$m\ddot{x} = -kx$$

using complex numbers (solution in your text)

Assume force is proportional to velocity and opposite in direction

$$f_{viscous} = -bv$$

(e.g. neglecting turbulence)

Newton's 2nd law:

$$F_{net} = F_{spring} + f_{viscous} = ma = m\ddot{x}.$$

So, $m\ddot{x} = -kx - bv$

or

$$m\ddot{x} + b\dot{x} + kx = 0$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0$$

Recall, $\omega_0 = \sqrt{k/m}$ and introduce shorthand $\beta = \frac{b}{2m}$ so that

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Call these

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}, \quad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Important differences in this solution depending on <u>regime</u> of the parameters β, ω_0.

IV <u>Three Regimes</u>

(1) Overdamped: $\beta > \omega_0$.

Solve for $x(t)$? Use standard guess, $x = e^{rt}$

$$\dot{x} = re^{rt}, \quad \ddot{x} = r^2 e^{rt}$$

$$r^2 \cancel{e^{rt}} + 2\beta r \cancel{e^{rt}} + \omega_0^2 \cancel{e^{rt}} = 0$$

$$\Rightarrow r^2 + 2\beta r + \omega_0^2 = 0$$

$$\Rightarrow r = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$$

$$= -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Then $B \equiv \sqrt{\beta^2 - \omega_0^2}$ is real and the general solution is

$$x(t) = C_1 e^{(-\beta + B)t} + C_2 e^{(-\beta - B)t}$$

Both terms are damping exponentials (Note $\beta > B > 0$).

[Aside: For exp.s the "characteristic time", τ, is the time it takes to get to $1/e$ of its original value.]

In the overdamped regime there are two characteristic times

$$\tau_1 = \frac{1}{\beta - B} \; ; \quad \tau_2 = \frac{1}{\beta + B}.$$

In fact, for Huge damping

$\beta \approx B$ and $\tau_1 \to \infty$.

$$= e^{-\beta t}\left(D_1 \cos\omega_1 t + D_2 \sin\omega_1 t\right)$$

with $D_1 = C_1 + C_2$, $D_2 = i(C_1 - C_2)$

$$\boxed{x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)}$$

Sinusoidal oscillations w/ exp. decreasing amplitude $A e^{-\beta t}$

(2) Underdamped: $\beta < \omega_0$.

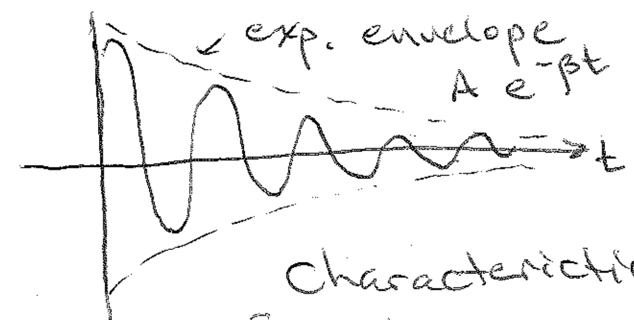
Then $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ is real

$$r_{1,2} = -\beta \pm i\omega_1$$

and the general soln. is

$$x(t) = C_1 e^{(-\beta + i\omega_1)t} + C_2 e^{(-\beta - i\omega_1)t}$$

$$= e^{-\beta}\Big(C_1[\cos\omega_1 t + i\sin\omega_1 t] + C_2[\cos\omega_1 t - i\sin\omega_1 t]\Big)$$



Characteristic time for damping $\tau = 1/\beta$

Two times at work here: damping $\tau = 1/\beta$; and the period of the oscillations

$$T = \frac{2\pi}{\omega_1}$$

<u>Question</u>: How many osc.s in one characteristic time?

$$\# = \frac{\tau}{T} = \frac{1}{\beta} \cdot \frac{\omega_1}{2\pi} = \frac{1}{\pi}\left(\frac{\omega_1}{2\beta}\right)$$

Quality factor $\equiv Q \equiv \omega_1 / 2\beta$

"How many times does it ring?"

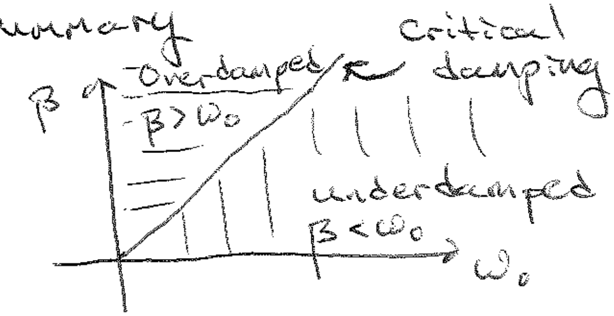
(3) <u>Critical Damping</u>: $\beta = \omega_0 \Rightarrow r_1 = r_2$

General Soln. is now $x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$

$$= e^{-\beta t}(C_1 + C_2 t)$$

Again $\tau = 1/\beta$

Summary



The derivatives in our damped osc. EOM can be collected into a <u>differential operator</u>:

$$D = \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2$$

so

$$Dx = \ddot{x} + 2\beta \dot{x} + \omega_0^2 x$$

This op. is linear

$D(ax) = aDx$ and $D(x_1 + x_2) = Dx_1 + Dx_2$

Linear op.s w/ constant coef.s are the ones where the standard guess works. Show it!