Mechanics

Day 20

Today's Outline

I. Last Lecture

II. Free-fall with $\vec{F}_{cor}$

III. Foucault's pendulum

I. Last Lecture

We carefully worked out:

- the direction of the Centrifugal force
- the impact of $\vec{F}_{cf}$ on $\vec{g}$.
- The direction of the coriolis force (in both the Northern and the Southern hemispheres).

II Free-fall again

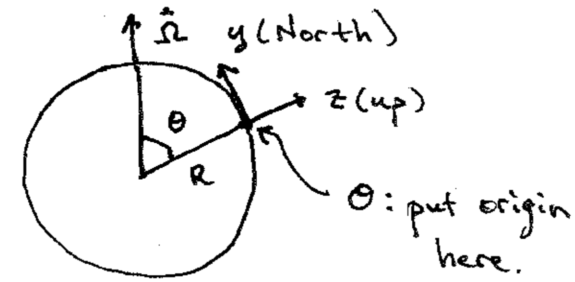
$$m\ddot{\vec{r}} = m\vec{g}_0 + \vec{F}_{cf} + \vec{F}_{cor}$$

$$= m\vec{g} + 2m\dot{\vec{r}} \times \vec{\Omega}$$

$$\Rightarrow \ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}} \times \vec{\Omega}$$

Notice that this equation only depends on $\ddot{\vec{r}}$ and $\dot{\vec{r}}$ $\Rightarrow$ we can arbitrarily shift our origin (see figure):

Cross-section of Earth:



In these coords

$$\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z}) \quad \text{and}$$

$$\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta), \text{ so that}$$

$$\dot{\vec{r}} \times \vec{\Omega} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & \Omega S\theta & \Omega C\theta \end{vmatrix} = (\dot{y}\Omega C\theta - \dot{z}\Omega S\theta, -\dot{x}\Omega C\theta, \dot{x}\Omega S\theta)$$

We found: E.O.M.s

$$\ddot{x} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$$
$$\ddot{y} = -2\Omega\dot{x}\cos\theta$$
$$\ddot{z} = -g + 2\Omega\dot{x}\sin\theta$$

which are coupled. We used successive approximations:

0^{th} order (Ω^0): $x=0$, $y=0$, $z=h-\frac{1}{2}gt^2$

Then

$$\ddot{x} = 2\Omega g t\sin\theta, \quad \ddot{y}=0, \quad \ddot{z}=-g$$

$$\Rightarrow \dot{x}(t) = \Omega g t^2\sin\theta + \cancelto{0}{v_{x0}}$$

$$\Rightarrow x(t) = \frac{1}{3}\Omega g t^3\sin\theta + \cancelto{0}{x_0}$$

So 1^{st} order is

(Ω^1): $x(t) = \frac{1}{3}\Omega g t^3 s\theta$; $y=0$; $z=h-\frac{1}{2}gt^2$

We could continue in this manner to get as many powers of Ω as we wanted. How big is this effect?

Drop a pebble down a 100 meter mine shaft at the equator and

we have $z = h-\frac{1}{2}gt^2 \Rightarrow t=\sqrt{\frac{2h}{g}}$

while

$$x = \frac{1}{3}\Omega g\left(\frac{2h}{g}\right)^{3/2} \approx 2.2\text{ cm}$$

generally a small effect.

- Successive approx is nice technique keep it in your tool box.

<u>Survey</u>: 1. How is the pace of the class on the whole? (Written feedback helpful 0 = too slow, 5 = just right, 10 = too fast)

2. What topic is any, is most unclear to you still? clearest? Why for both?

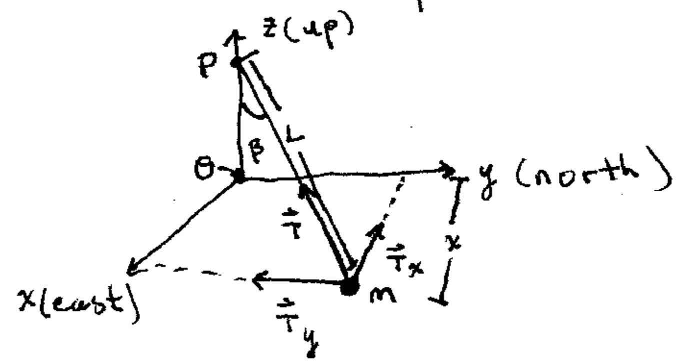
3. Which aspect of class has been most effective for you? Least?

4. What suggestions do you have for improving the class overall?

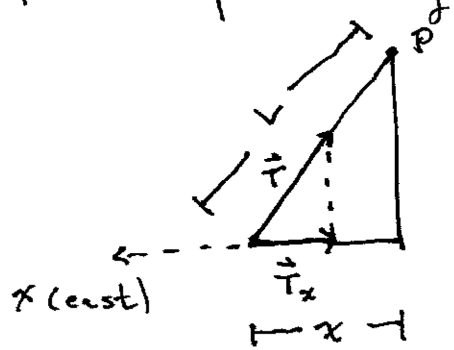
5 Other comments or suggestions?

III Foucault's pendulum

Foucault's pendulum is a spherical pendulum (like your HW problem) with a massive bob and a long wire. The pendulum is suspended from a pivot P:



Now lets look at our figure in the plane spanned by $\vec{T}_x$ and $\vec{T}$:



By similar triangles we have

$$\frac{T_x}{T} = -\frac{x}{L}$$

Minus sign means T_x points in neg. x-direction.

The E.O.M. for the pendulum is

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g}_0 + \vec{F}_{cf} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

$$= \vec{T} + m\vec{g} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

Consider case where β is small so that

$$T_z = T\cos\beta \approx T$$

For small β we also have $\ddot{z}$ and $\dot{z}$ small and the z-component of the E.O.M. becomes,

$$0 = T_z - mg \Rightarrow T_z \approx T \approx mg.$$

Similarly $T_y = -T\, y/L = -mgy/L$

Putting it together our E.O.M are

$$\ddot{x} = -gx/L + 2\dot{y}\,\Omega\cos\theta$$

$$\ddot{y} = -gy/L - 2\dot{x}\,\Omega\cos\theta$$

Noting that $g/L = \omega_0$ and $\Omega\cos\theta = \Omega_z$ we have

$$\ddot{x} - 2\Omega_z\dot{y} + \omega_0^2 x = 0$$

$$\ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 y = 0$$

Another set of coupled equations!

These are almost harmonic oscillator equations. We'll use another new technique to solve them. Let

$$\eta = x + iy$$

and multiply the $\ddot{y}$ equation by $i = \sqrt{-1}$ and add it to the $\ddot{x}$ equation, to find

$$\ddot{\eta} + 2i\Omega_z \dot{\eta} + \omega_0^2 \eta = 0$$

This is a 2nd order, linear, homogeneous diff. eq. We can go back to our

Standard guess:

$$\eta(t) = e^{\lambda t}$$

(Note: notation differs from book but agrees with lecture 1.)

$$\Rightarrow \lambda^2 + 2i\Omega_z \lambda + \omega_0^2 = 0$$

$$\Rightarrow \lambda = \frac{-2i\Omega_z \pm \sqrt{-4\Omega_z^2 - 4\omega_0^2}}{2}$$

$$= -i\left(\Omega_z \mp \sqrt{\Omega_z^2 + \omega_0^2}\right)$$

(so book's α = our $i\lambda$).

Recall $\Omega_z \ll \omega_0$ and so

$$\lambda \approx -i(\Omega_z \mp \omega_0)$$

Our general solution is then

$$\eta(t) = c_1 e^{-i(\Omega_z - \omega_0)t} + c_2 e^{-i(\Omega_z + \omega_0)t}$$

$$= e^{-i\Omega_z t}\left[c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}\right]$$

To set c_1 and c_2 we need initial conditions, let's choose: $x(0) = A$ $y(0) = 0$

$v_{x_0} = v_{y_0} = 0$ then $\eta(0) = A$ and $\dot{\eta}(0) = 0$ but

$$\eta(0) = c_1 + c_2$$

$$\dot{\eta}(0) = -i(\Omega_z - \omega_0)c_1 - i(\Omega_z + \omega_0)c_2$$

$$\approx i\omega_0 c_1 - i\omega_0 c_2$$

Then, $c_1 + c_2 = A$ $c_1 - c_2 = 0$

$$\Rightarrow c_1 = c_2 = A/2.$$

and

rotates plane of oscillation

$$\eta(t) = x + iy = A e^{-i\Omega_z t} \cos(\omega_0 t)$$

usual Pendulum motion