Mechanics
Day 21

Today

I Survey results

II Last time / where are we?

III Collections of particles

IV Why must we distinguish $\vec{\omega}$ and $\vec{L}$?

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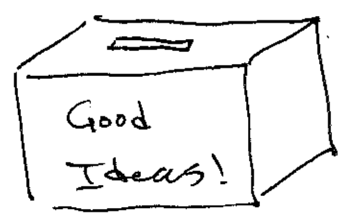
Perhaps a tiny bit too fast. I'll try to make appropriate adjustment.

You were also interested in doing more problem solving together. I like this idea and will try to schedule some problems specifically for office hours.

Finally, another suggestion that a few of you put forward was to do some more demos. I'm excited about this and hope that some of you will volunteer to help me build some demos.

Your feedback on the course is always welcome. I appreciate your ideas on how to make the course as valuable as possible.



Any additional comments:

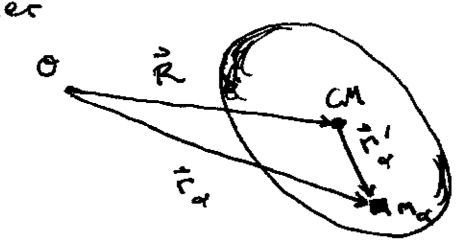
II Last time / Where are we?

We investigated rotating frames for two reasons:

1. To understand motion in noninertial frames: Modify Newton's 2nd law to ~~take~~ include fictitious forces.
2. To setup context for investigating rotating bodies (as opposed to frames).

This formula is amazing (!): We can treat collections of particles as a single pt. particle with mass M subject to $\vec{F}^{ext}$.

Consider



a rigid body and define $\vec{r}_\alpha'$ as the position of m_α w.r.t. the C.M.

III Collections of particles

CM: $\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha = \frac{1}{M} m_\alpha \vec{r}_\alpha$ $(\alpha = 1, \dots, N)$

Einstein summation convention: repeated index means sum over that index.

Total Momentum: $\vec{P} = \sum_\alpha p_\alpha = m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$.
$\left(= \frac{d}{dt}(m_\alpha \vec{r}_\alpha) = \frac{d}{dt}(M\vec{R})\right)$

Total external Force: $\dot{\vec{P}} = \vec{F}^{ext} = M\ddot{\vec{R}}$

[Aside: Throughout this chapter Taylor forgoes integration unless it is absolutely necessary. A nice convention but do be careful.]

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha'$$

Ang. mom. of m_α: $\vec{l}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha = \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha$ (no sum)

Tot. Ang. mom.: $\vec{L} = \sum_\alpha \vec{l}_\alpha = \sum_\alpha \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha$ ← this simplifies

$= \sum_\alpha (\vec{R} + \vec{r}_\alpha') \times m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}_\alpha')$

So,

$$\vec{L} = \sum \vec{R} \times m_\alpha \dot{\vec{R}} + \sum \vec{R} \times m_\alpha \dot{\vec{r}}'_\alpha + \sum \vec{r}'_\alpha \times m_\alpha \dot{\vec{R}} + \sum \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha$$

$$= \vec{R} \times M\dot{\vec{R}} + \vec{R} \times \Big(\underbrace{\sum m_\alpha \dot{\vec{r}}'_\alpha}_{0\text{, time derivative of zero}}\Big) + \Big(\underbrace{\sum \vec{r}'_\alpha m_\alpha}_{0\text{, CM position relative to CM}}\Big) \times \dot{\vec{R}} + \sum \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha$$

$$= \vec{R} \times \vec{P} + \sum \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha$$

$$\Rightarrow \vec{L} = \vec{L}(\text{motion of CM}) + \vec{L}\left(\begin{matrix}\text{motion}\\ \text{relative to CM}\end{matrix}\right)$$

For example a planet orbiting the sun (assume sun fixed):

$$\vec{L} = \underbrace{\vec{L}_{orb}}_{\text{orbital angular momentum due to motion around the sun}} + \underbrace{\vec{L}_{spin}}_{\text{spin ang. mom. due to motion relative to CM.}}$$

Total Kinetic Energy: $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}$

Also simplifies:

$$\dot{\vec{r}}_\alpha^{\,2} = (\dot{\vec{R}} + \dot{\vec{r}}'_\alpha)^2 = \dot{\vec{R}}^2 + \dot{\vec{r}}'^2_\alpha + 2\dot{\vec{R}}\cdot\dot{\vec{r}}'_\alpha$$

$$\Rightarrow T = \frac{1}{2}\sum m_\alpha \dot{\vec{R}}^2 + \frac{1}{2}\sum m_\alpha \dot{\vec{r}}'^2_\alpha + \dot{\vec{R}}\underbrace{\sum m_\alpha \dot{\vec{r}}'_\alpha}_{\text{zero as before}}$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} m_\alpha \dot{\vec{r}}'^2_\alpha$$

In words

$$T = T(\text{motion of CM}) + T\left(\begin{matrix}\text{motion relative}\\ \text{to CM}\end{matrix}\right)$$

For a rigid body any motion relative to the CM is a rotation (Euler's theorem)

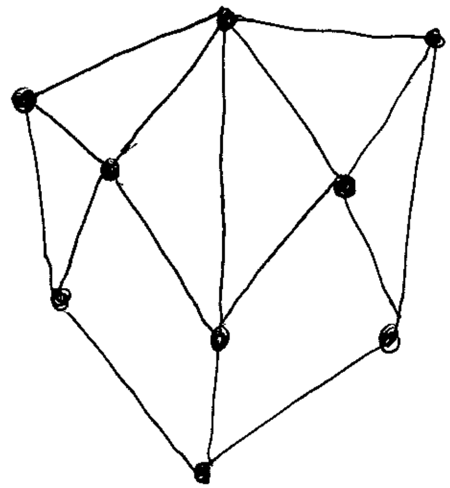
and so

$$T = T\left(\begin{matrix}\text{motion of}\\ \text{CM}\end{matrix}\right) + T\left(\begin{matrix}\text{rotation}\\ \text{about CM}\end{matrix}\right)$$

Finally, if the forces on and within a body are conservative,

$$U = \underbrace{U^{ext}}_{\text{total external potential energy}} + \underbrace{U^{int}}_{\text{gives rise to forces that hold body together}}$$

$$U^{int} = \sum_{\alpha<\beta} U_{\alpha\beta}(\underbrace{r_{\alpha\beta}}_{\text{dist. btwn particles } \alpha \text{ and } \beta}) = \text{constant we can drop.}$$



Example of a rigid body with fixed interparticle spacings.

$$T = \tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2} m_\alpha \dot{\vec{r}}_\alpha'^2$$

← Einstein summation

or

$$T = T(\text{motion of CM}) + T\left(\begin{matrix}\text{motion}\\ \text{relative to CM}\end{matrix}\right)$$

or for a rigid body

$$T = T(\text{motion of CM}) + T(\text{rotation } \cancel{\text{about}} \text{ CM})$$

and

$$U = U^{ext} + U^{int}$$ ← for rigid body w/ conservative internal forces, a constant that can be dropped.

Let us briefly tabulate all of these results:

$$\vec{P} = M\dot{\vec{R}} \qquad \left(\begin{matrix}\vec{R} = \text{CM position}\\ M = \text{total mass}\end{matrix}\right)$$

$$\dot{\vec{P}} = \vec{F}^{ext} = M\ddot{\vec{R}}$$

$$\vec{L} = \vec{R}\times\vec{P} + \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha'$$

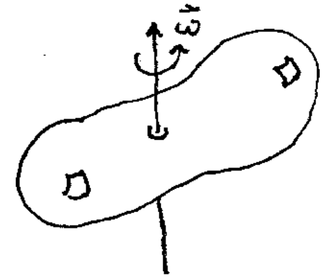
position relative to CM $(\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha')$

or

$$\vec{L} = \vec{L}(\text{motion of CM}) + \vec{L}\left(\begin{matrix}\text{motion}\\ \text{relative to CM}\end{matrix}\right).$$

IV Why must we distinguish $\vec{\omega}$ and $\vec{L}$?

<u>Example</u>: Fixed axis of rotation.



"speared potato"

Choose fixed axis to be z axis:

$$\vec{\omega} = (0, 0, \omega)$$

We want to calculate

$$\vec{L} = \sum \vec{l}_\alpha = \sum \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

Well,

$$\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$$

and $\vec{v}_\alpha = (\vec{\omega} \times \vec{r}_\alpha)$ (recall $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$)

$$\vec{v}_\alpha = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = (-y_\alpha \omega, x_\alpha \omega, 0)$$

$$l_\alpha = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_\alpha & y_\alpha & z_\alpha \\ -m_\alpha y_\alpha \omega & m_\alpha x_\alpha \omega & 0 \end{vmatrix} = m_\alpha \omega (-x_\alpha z_\alpha, -y_\alpha z_\alpha, x_\alpha^2 + y_\alpha^2) \quad \text{(no sum)}.$$

Then

$$L_z = \sum m_\alpha (x_\alpha^2 + y_\alpha^2)\, \omega$$

$$= m_\alpha \rho_\alpha^2 \omega = I_z \omega$$

with $I_z = m_\alpha \rho_\alpha^2$. This may look familiar. But

$$L_x = -\sum \underbrace{m_\alpha x_\alpha z_\alpha}\, \omega$$

$$L_y = -\sum \underbrace{m_\alpha y_\alpha z_\alpha}\, \omega$$

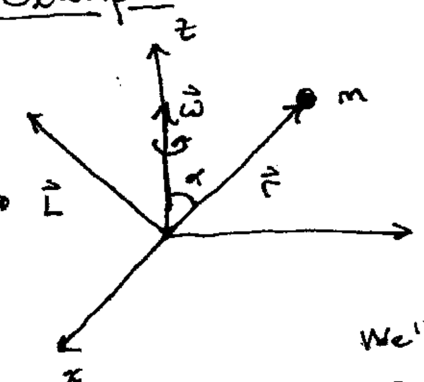
are new. These prefactors are called the "products of inertia"

So, why must we distinguish $\vec{\omega}$ and $\vec{L}$? [The answer is even richer than why we distinguish $\vec{v}$ and $\vec{p}$!] Unlike $\vec{v}$ and $\vec{p}$ they don't just differ by a scalar ($\vec{p} = m\vec{v}$) but they can even point in different directions (!!!):

$\vec{\omega}$ is not always parallel to $\vec{L}$.

Why? $\vec{L}$ depends on how the mass is distrubuted around the rotation axis, that is, on the body's shape.

<u>Another example:</u>



$\vec{L}$ precesses around $\vec{\omega}$

We'll return to this example.