Classical Mechanics

Day 3

Sep 4th, 2015 p1/4

<u>Outline</u>

I Homogeneous linear differential equations

II Last ~~Re~~ Meeting ~~&~~ damped oscillator wrap up

III Driven damped oscillator

$$a_n \frac{d^n x}{dt^n} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0.$$

Any eqn. like this can be solved by x of the form:

$$x(t) = e^{rt}$$

w/ r const. (our standard guess).

<u>why it works</u>: $x = e^{rt} \Rightarrow \frac{dx}{dt} = r e^{rt}$

$$\frac{d^2x}{dt^2} = r^2 e^{rt}, \ldots, \frac{d^n x}{dt^n} = r^n e^{rt}$$

I Homog. linear eqn.s w/ constant coeff.:

<u>Linear</u>: $a_n(t) \frac{d^n x}{dt^n} + a_{n-1}(t) \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1(t) \frac{dx}{dt} + a_0(t) x = f(t)$

<u>Homogeneous</u>: $f(t) = 0$.

<u>Constant coeff.</u>: $a_0, a_1, \ldots, a_n$ do <u>not</u> depend on t.

So,

$$a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \cdots + a_0 e^{rt} = 0$$

$$\Rightarrow a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0 \quad (\text{"roots"})$$

Fundamental thm. of Algebra guarantees that there are n soln.s to Eq. (root).

<u>clean up</u>: Collect the a_i and the deriv.s into a <u>differential</u>

Operator:

$$D \equiv a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{d}{dt} + a_0$$

Ex.: damped oscillator

$$D \equiv \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2$$

$$D x = \ddot{x} + 2\beta \dot{x} + \omega_0^2 x$$

This operator is linear

$$D(ax) = a Dx, \quad D(x_1 + x_2) = Dx_1 + Dx_2$$

with the r_i solutions of Eq. (roots).

Except: if the algebraic eqn. gives multiple roots, then I'm missing solutions. S'pose

$$\left(\frac{d}{dt} - r\right)\left(\frac{d}{dt} - r\right) x = 0$$

Let $y = \left(\frac{d}{dt} - r\right) x$ and so

$$\left(\frac{d}{dt} - r\right) y = 0$$

$$\Rightarrow \quad y = C e^{rt}$$

For linear & homog. diff. eqn.s any linear combo of solns is a soln because

$$D(a x_1 + b x_2) = a D x_1 + b D x_2$$
$$= 0 + 0 = 0. \checkmark$$

For n^{th} order eqn.s the linear combo w/ n constants is the general solution

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \cdots + C_n e^{r_n t}$$

But, then

$$\left(\frac{d}{dt} - r\right) x = C e^{rt}$$

Check $x = Ct e^{rt}$ solves it

$$\left(C e^{rt} + C \cancel{t} r e^{rt} - r \cancel{Ct e^{rt}}\right) \stackrel{?}{=} C e^{rt}$$

yes! In general, for an m times repeated root, soln.s are

$$e^{rt}, \; t e^{rt}, \; t^2 e^{rt}, \; \ldots, \; t^{m-1} e^{rt}$$

Last time: damped oscillator

$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

w/ $r_1 = -\beta + \sqrt{\beta^2 + \omega_0^2}$, $r_2 = -\beta - \sqrt{\beta^2+\omega_0^2}$

$\beta = \frac{b}{2m}$ ← damping coeff., $\omega_0 = \sqrt{\frac{k}{m}}$, $B \equiv \sqrt{\beta^2 - \omega_0^2}$

$\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$.

Started to explore three regimes

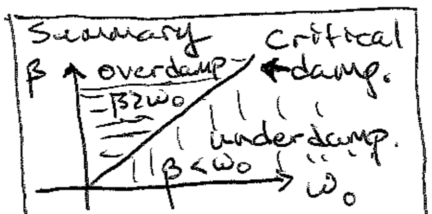
(1) Overdamped: $\beta > \omega_0$. Exponentially

Also two times at work here:

characteristic damping time $\tau_1 = 1/\beta$;

and period of oscillation

$$\tau_2 = \frac{2\pi}{\omega_1}$$



Question: How many oscillations in one characteristic time?

$$\# = \frac{\tau_1}{\tau_2} = \frac{1}{\beta \frac{2\pi}{\omega_1}} = \frac{1}{\pi}\frac{\omega_1}{2\beta} \approx \frac{1}{\pi}\frac{\omega_0}{2\beta}$$

small damping or large mass

quality factor $Q \equiv \frac{\omega_0}{2\beta}$

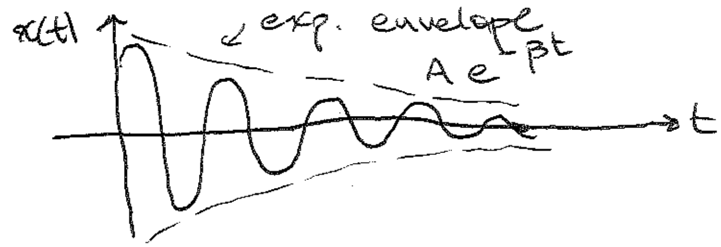
damped motion with two characteristic times

$$\tau_1 = \frac{1}{\beta - B}, \quad \tau_2 = \frac{1}{\beta + B}$$

(2) Underdamped: $\omega_0 > \beta$

showed that

$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$



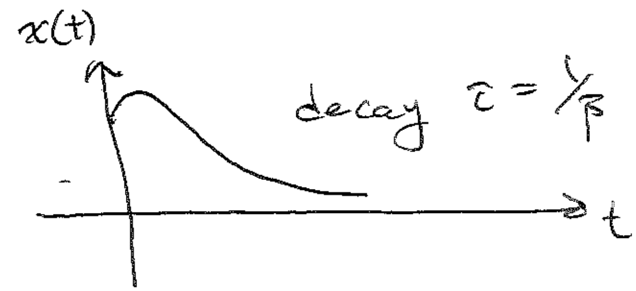
(3) Critical damping: $\beta = \omega_0$

$\Rightarrow r_1 = r_2$ multiple root

General solution

$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$

$= e^{-\beta t}(C_1 + C_2 t)$



III <u>Driven damped oscillator</u>

Think of a child on a swing: oscillator (swing), damping (friction) driving (you pushing the child)

$$F_{net} = -kx - bv + F(t)$$

In general the driving depends on time.

So, $m\ddot{x} + b\dot{x} + kx = F(t)$

$$\Rightarrow \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \equiv \frac{F(t)}{m} \text{ (drive)}$$

ω_0 the "natural freq." and

ω the "driving freq.".

This is an example of an inhomogeneous linear diff. eqn.

S'pose $x_h(t)$ is the general soln. to the associated homog. eqn then

$$\overset{\swarrow \text{diff. op.}}{D}\, x_h = 0$$

Small f is the force per unit mass. We'll be particularly interested in

$$f(t) = f_0 \cos(\omega t)$$

('cause it's physically reasonable <u>and</u> soluable — can build more complicated driving out of cosines too). Note well the distinction btwn

$$D \equiv \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

S'pose further that $x_p(t)$ is any old ("particular") soln. to the inhomog. eqn:

$$D\, x_p(t) = f(t)$$

Then $x_h + x_p$ solves the inhomog. eqn.

$$D(x_h + x_p) = Dx_h + Dx_p = 0 + f(t)$$

and is, in fact, the general soln.