Classical Mechanics

Day 4

Sep 7th, 2015 P1/4

<u>Outline</u>

I Driven damped oscillator

II Resonance

I We found the E.O.M.:

call it → (driven) $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \equiv \frac{F(t)}{m}$

Decided to focus on drive

$$f(t) = f_0 \cos(\omega t)$$

Recall,

$\omega_0 =$ natural freq.

$\omega =$ driving freq.

We realized

$$D(x_h + x_p) = Dx_h + Dx_p$$
$$= 0 + f(t)$$

where

$$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

$x_h =$ general soln. of inhomog. eqn.

$x_p =$ "particular soln."

[On the homework you will show it doesn't matter what x_p you use.]

So to solve (driven) we focus on finding a particular soln.

Do this by hook or by crook!

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

Exponentials are nice, so view this as the real part of

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

where $z(t) \equiv x(t) + iy(t)$.

Then the imaginary part is

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin\omega t.$$

Try to guess a particular soln

$$z_p = C e^{i\omega t}$$

Physically this says that if we drive an osc. at freq. ω we might expect it to respond at the same freq. ω. Then

$$\dot{z}_p = i\omega C e^{i\omega t} \quad \text{and} \quad \ddot{z} = -\omega^2 C e^{i\omega t}$$

and

$$-C\omega^2 e^{i\omega t} + 2\beta i\omega C e^{i\omega t} + \omega_0^2 C e^{i\omega t} = f_0 e^{i\omega t}$$

so,

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2\beta i\omega}$$

We got a particular soln! We'll simplify it next. Nice to write

$$z_p = C e^{i\omega t} = \underset{\uparrow \text{ real}}{A} e^{-i\delta} e^{i\omega t}$$

Recall,

$$|e^{i\theta}|^2 = e^{i\theta} \cdot e^{-i\theta} = 1$$

so,

$$|A|^2 = |C|^2 = C \cdot C^* \leftarrow \text{complex conj.}$$

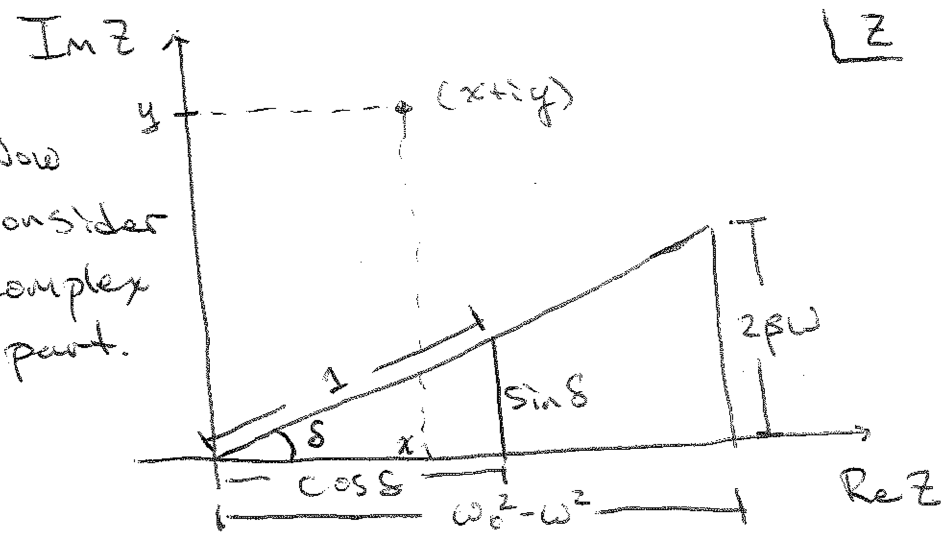
$$= \frac{f_0}{(\omega_0^2 - \omega^2) + 2\beta i\omega} \cdot \frac{f_0}{(\omega_0^2 - \omega^2) - 2\beta i\omega}$$

$$= \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

This implies that

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

Now consider complex part.



So, $C = Ae^{-i\delta}$ gives

$$e^{i\delta} = \frac{A}{C} = \frac{A}{f_0}\left(\omega_0^2 - \omega^2 + 2\beta i\omega\right)$$

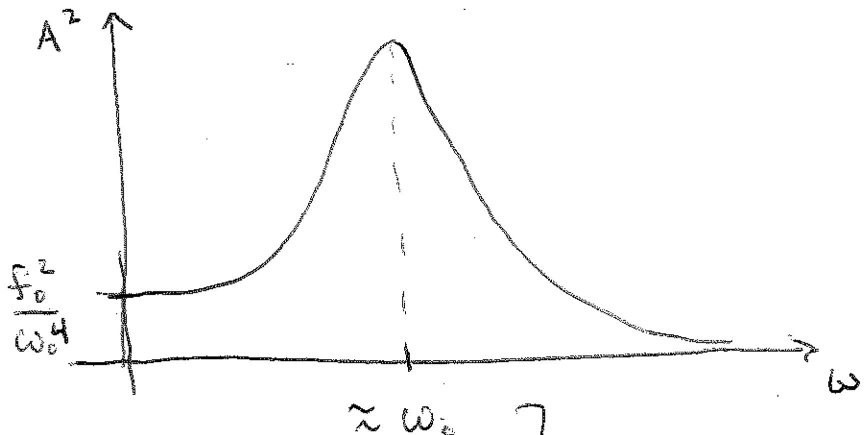
From the diagram

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

This gives us a way to find δ.

So our general solution is (Real part)

$$x(t) = \overset{\text{not arbitrary}}{A}\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$$



$$\left.A^2_{max} \approx \frac{f_0^2}{4\beta^2\omega_0^2}\right\}\text{ show this:}$$

A^2 is max when the denominator is minimized, so it is near $\omega = \omega_0$, but

with the latter both exponentially damped <u>transients</u>. After transients have died out the motion is independent of initial conditions and the $A\cos(\omega t - \delta)$ term is called an <u>attractor</u>.

<u>II Resonance</u>: Analyze the amplitude A: $A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$

let's calculate it:

$$\frac{d}{d\omega}\left((\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right) = 0$$

$$\Rightarrow 2(\omega_0^2 - \omega^2)\cdot(-2\omega) + 8\beta^2\omega = 0$$

$$\Rightarrow \omega^2 - \omega_0^2 + 2\beta^2 = 0$$

$$\Rightarrow \omega = \sqrt{\omega_0^2 - 2\beta^2} \equiv \overset{\text{call it}}{\omega_2}$$

Indeed, for small β

$$\omega_2 \approx \omega_0$$

Then,

$$A_{max}^2 \approx A^2(\omega_0) = \frac{f_0^2}{4\beta^2\omega_0^2}$$

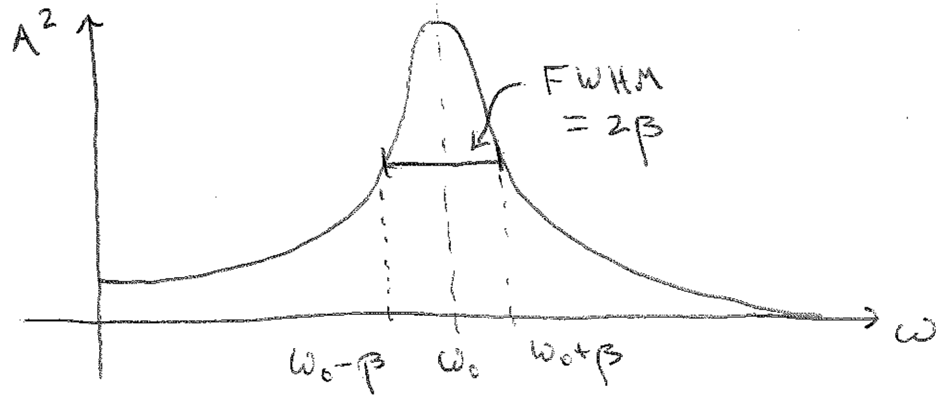
We've characterized the height of the resonance. What about its width? This is done with:

FWHM = "Full width at Half Max"

or

HWHM = "Half Width at Half Max"

Again assume $(\beta \ll \omega_0)$ β is small



A dimensionless measure of the sharpness of the resonance is the <u>Quality factor</u>

$$Q = \frac{\omega_0}{2\beta}$$

Large value of $Q \Rightarrow$ narrow resonance.

Then for $\omega = \omega_0 \pm \beta$,

$$\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega_0(\mp\beta)$$

and

$$4\beta^2\omega^2 \approx 4\beta^2\omega_0^2 + O(\beta^3)$$

So,

$$A^2(\omega_0 \pm \beta) \approx \frac{f_0^2}{(\mp 2\beta\omega_0)^2 + 4\beta^2\omega_0^2} = \frac{f_0^2}{8\beta^2\omega_0^2} \approx \frac{1}{2} A_{max}^2$$

<u>Examples of resonance:</u>

Electrical: Many, many, but e.g. a radio tuner.

Optical spectra: e.g. the absorption spectrum of the sun.

Nuclear: NMR = Nuclear Magnetic Resonance — good for medical imaging.