Mechanics
Day 5

Sept 9th, 2015

<u>Outline</u>

I Motivation

II Chain Rule

III Euler's Method in the Calculus of Variations

I. New approach to Mechanics called a variational principle. Based on new mathematics: the calculus of variations.

Because this approach is new and its tools are new, it can be easy to lose the forest for the trees. This motivation will try to mitigate this.

Recall that Newton's laws are a set of axioms:

I. Velocity is const. unless body is acted on by a force.

II. The acceleration is given by
$$\vec{a} = \vec{F}/m$$

III Mutual forces of action and reaction are equal, opposite and collinear.

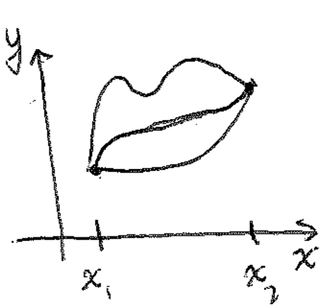
They have been promoted to the status of laws because of their incredible predictive success.

But, <u>other</u> axioms are possible.

A variational principle expresses the idea that the correct motion of a system can be predicted by extremizing (minimizing, maximizing, saddling) an integral:

e.g. $I = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$

Eventually the integral itself will take on physical meaning (particularly in Quantum Mechanics), but for now we will focus on the path $y(x)$ that we feed to the integral. By changing this path we change the value of I and our immediate goal will be



Different paths give different integrals

to find a condition on the path that guarantees it will extremize I.

Physically our independent variable will be t and our path will be, e.g. $x(t)$. We will find an integral $S[x(t)]$ that is extremized when $x(t)$ satisfies the physical "equations of motion" E.O.M. This is our goal.

<u>Strategy</u>: Math first then physics.

<u>Why?</u>: Allows us to separate the calculation and its interpretation.

<u>Notation</u>: I for integral

x for independent variable

y for dependent variable

$y(x)$ for the path

} Appropriate for math

We will adopt physical notation starting next week.

Why <u>bother with all of this?</u>

Many reasons, but one important one is that it will free us from the shackles of Cartesian coord.s

Adapt our coord.s to the symmetry of the system. I'll list more as we go.

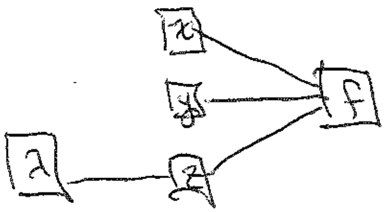
Today: Euler's method (not in your book)

Friday: Lagrange's Method (in your book)

III Consider $f = f(x, y, z)$ and the case in which $z = z(\lambda)$, so that

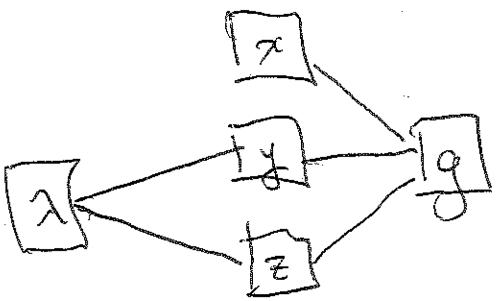
$$f = f(x, y, z(\lambda))$$

Schematically



If we shake (that is, vary) y then $\frac{\partial f}{\partial y}$

<u>Two subtleties</u> (1) What if $g = g(x, y(\lambda), z(\lambda))$?

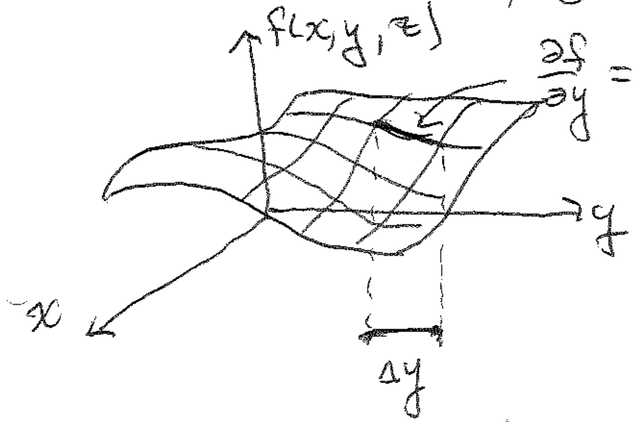


$$\frac{d}{d\lambda}(g) = \frac{\partial g}{\partial y}\frac{dy}{d\lambda} + \frac{\partial g}{\partial z}\frac{dz}{d\lambda}$$

↑ you add contributions

If $\frac{dy}{d\lambda} = 0$ recovers previous example

gives how f changes when you vary y keeping all other variables fixed, graphically



$\frac{\partial f}{\partial y}$ = slope of tangent line

What about if you shake λ? Causes z to shake and $\frac{df}{d\lambda} = \frac{\partial f}{\partial z}\frac{dz}{d\lambda}$

(2) What if $f = f(x, y, z(\lambda))$ and we know $z(\lambda) = \lambda^2$?

You might like to write

$$f = f(x, y, z(\lambda)) = f(x, y, \lambda^2)$$

But $\frac{d(f)}{d\lambda} = \frac{\partial f}{\partial \lambda^2}\frac{d\lambda^2}{d\lambda}$ (wrong eqn!)

Instead $\frac{d(f)}{d\lambda} = \frac{\partial f}{\partial z}\frac{dz}{d\lambda} = \frac{\partial f}{\partial z}\cdot(2\lambda)$.

Sometimes we write $D_3 f(x, y, z) \equiv \frac{\partial f}{\partial z}$. to avoid this confusion.

III Euler's Method

s'pose that the curve in Figure 1 extremizes the integral

$$I = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$$

We want to find an equation that determines this curve $y(x)$.

We will proceed in an approximate manner, similar to a Riemann sum in calculus:

(1) Divide the interval btwn $x = x_1$ and $x = x_2$ into many subintervals of width Δx.

(2) Approximate the integral by a sum

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \simeq \sum_{x=x_1}^{x_2} f(x, y, y')\, \Delta x$$

In each term of this sum evaluate f at the initial pt, e.g. x_N, $y(x_N) = y_n$ for interval $[x_N, x_O]$.

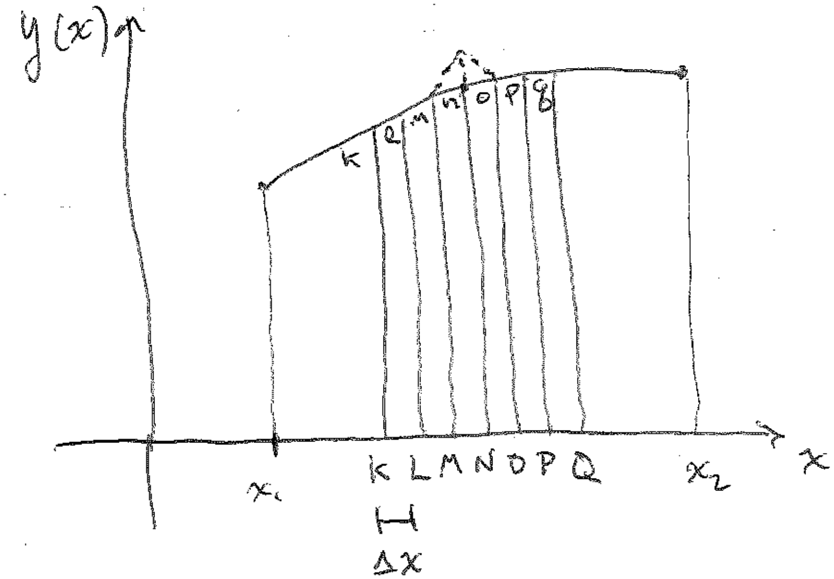


Figure 1

(3) Approximate the derivative $y' = \frac{dy}{dx}$ by the slope of the straight line connecting the end pts of the interval. That is,

$$y'_m \equiv \frac{\Delta y}{\Delta x} = \frac{y_n - y_m}{\Delta x} \simeq y'(x_M)$$

All of these approximations become excellent in the limit of many subintervals, ie. as $x \to 0$.

Now, let's change y_n (see dashed lines in Fig. 1). This changes the sum, but only the terms involving y_n. The sum no longer depends on the whole path just the values of f at the discrete pts, x_n, y_n and y_n'. So, we can extremize it with regular Calculus:

$$\frac{\partial}{\partial y_n}\left(\sum_{x=x_1}^{x_2} f(x,y,y')\,\Delta x\right) = 0.$$

Setting this equal to zero and rearranging gives,

$$\frac{\partial f}{\partial y_n}\cdot\Delta x - \left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_m'}\right) = 0$$

Dividing by Δx,

$$\frac{\partial f}{\partial y_n} - \frac{1}{\Delta x}\left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_m'}\right) = 0$$

In the limit $\Delta x \to 0$ this is

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

The Euler-Lagrange Equation

Let's do it:

$$\frac{\partial}{\partial y_n}\left(\sum_{x=x_1}^{x_2} f(x,y,y')\,\Delta x\right)$$

terms not involving y_n

$$= \frac{\partial}{\partial y_n}\left(\cdots + f\left(x_m, y_m, \frac{y_n - y_m}{\Delta x}\right)\Delta x + f\left(x_n, y_n, \frac{y_o - y_n}{\Delta x}\right)\Delta x + \cdots\right)$$

$$\overset{\text{chain rule}}{=} \frac{\partial f}{\partial y_m'}\frac{\partial}{\partial y_n}\left(\frac{y_n - y_m}{\Delta x}\right)\cdot\Delta x + \frac{\partial f}{\partial y_n}\cdot\Delta x + \frac{\partial f}{\partial y_n'}\frac{\partial}{\partial y_n}\left(\frac{y_o - y_n}{\Delta x}\right)\cdot\Delta x$$

N.B.: Important distinction btwn partial and total derivatives here. t

Euler's method is nice bcse it is closely tied to the geometry. Next lecture we will look at Lagrange's method, which is mathematically slicker.