Mechanics
Day 6

<u>Outline</u>

I Last time

II Example of Euler's Method

III Lagrange's method in the calculus of variations

IV Example: Geodesics on Cylinder

We found

$$\frac{\partial f}{\partial y_n} - \frac{1}{\Delta x}\left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_m'}\right) = 0$$

as a condition for

$$\sum_{x=x_1}^{x_2} f(x, y, y')\,\Delta x$$

to be extremized w.r.t. variations of y_n. See Fig 1 for notation.

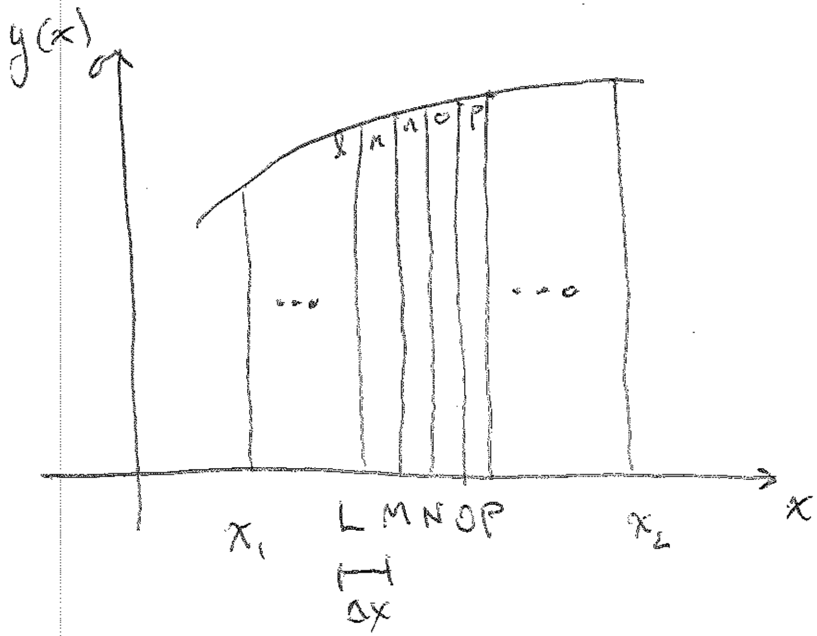


Figure 1

In the limit $\Delta x \to 0$ we have

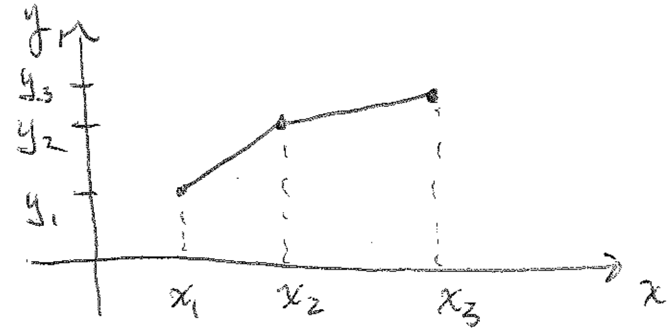
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

<u>Notice</u> that our derivations holds for any pt. of the curve. That is, an arbitrary x and $y(x)$, hence all of them. Thus, in using Euler's method you can always focus on a small segment of the curve.

II Let's do an example to illustrate this.

Q: What is the shortest path connecting two points in the xy-plane?

Focus first on (x_1, y_1) and (x_3, y_3)



The length of the path connecting them depends on our choice of (x_2, y_2)

This length is

$$L = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} + \sqrt{(x_3-x_2)^2+(y_3-y_2)^2}$$

$$= \sqrt{\Delta x^2+(y_2-y_1)^2} + \sqrt{\Delta x^2+(y_3-y_2)^2}$$

We want to choose y_2 such that L is a minimum.

So we impose,

$$\frac{\partial L}{\partial y_2} = 0$$

$$\Rightarrow \frac{1}{2}\frac{1}{\sqrt{\Delta x^2+(y_2-y_1)^2}} \cdot 2(y_2-y_1) + \frac{1}{2}\frac{1}{\sqrt{\Delta x^2+(y_3-y_2)^2}} \cdot 2(y_3-y_2)(-1) = 0$$

$$\Rightarrow \frac{y_2-y_1}{\sqrt{\Delta x^2+(y_2-y_1)^2}} = \frac{y_3-y_2}{\sqrt{\Delta x^2+(y_2-y_1)^2}}$$

Factor out Δx^2 to find

$$\frac{(y_2-y_1)/\Delta x}{\sqrt{1-\frac{(y_2-y_1)^2}{\Delta x^2}}} = \frac{(y_3-y_2)/\Delta x}{\sqrt{1-\frac{(y_3-y_2)^2}{\Delta x^2}}}$$

and we get equality if

$$\left(\frac{y_2-y_1}{\Delta x}\right) = \left(\frac{y_3-y_2}{\Delta x}\right)$$

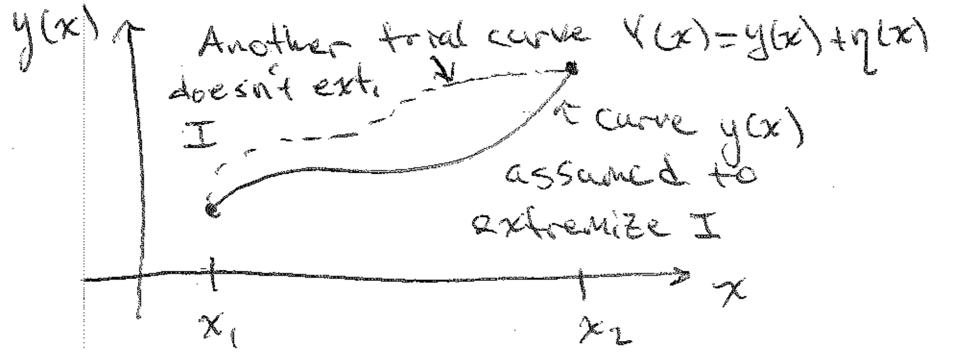
that is, if the slope is constant!

It's a straight line!!

III Lagrange's Method

We want to find the curve $y(x)$ that extremizes

$$I = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$$



Lagrange also wants to turn this into a calculus problem. He found a clever way to do it: Write your trial ~~pa~~ path as

$$Y(x) = y(x) + \alpha\,\eta(x)$$

in terms of a parameter α. Now, you can turn on a variation of the whole curve just by changing α. In particular, I becomes a normal function of α, $I(\alpha)$. So the condition that I is extremized is just

$$\frac{dI}{d\alpha} = 0$$

[Subtlety: We assume $y(x)$ is the "right" curve, so that we know where to evaluate, namely

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0.$$]

Let's do it, if $Y(x) = y(x) + \alpha\,\eta(x)$ then

$$Y'(x) = y'(x) + \alpha\,\eta'(x)$$

so, if

$$I(\alpha) = \int_{x_1}^{x_2} f(x, Y, Y')\, dx$$

$$= \int_{x_1}^{x_2} f(x, y(x) + \alpha\eta, y' + \alpha\eta')\, dx$$

Then, taking deriv. inside integral makes it a partial.

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\int_{x_1}^{x_2} f\, dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha}\, dx$$

Now, $\frac{\partial}{\partial \alpha}\left[f(x, y+\alpha\eta, y'+\alpha\eta') \right]$

chain rule

$$= \frac{\partial f}{\partial y}\cdot\frac{\partial}{\partial \alpha}(y+\alpha\eta) + \frac{\partial f}{\partial y'}\cdot\frac{\partial}{\partial \alpha}(y'+\alpha\eta')$$

$$= \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'$$

So,

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right)dx$$

"Can switch derivative at the cost of a minus sign and a boundary term"

Then,

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\frac{d\eta}{dx}\,dx = \underbrace{\left(\frac{\partial f}{\partial y'}\eta\right)\Big|_{x_1}^{x_2}}_{=0 \text{ bcse of assumed b.c. on } \eta} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\,dx$$

and

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta\,dx$$

To simplify the 2nd term use integration by parts

$$\int_{x_1}^{x_2}\frac{d}{dx}(f(x)g(x))\,dx = f(x)g(x)\Big|_{x_1}^{x_2}$$

(integral = antiderivative). But, also

$$\int_{x_1}^{x_2}\frac{d}{dx}(fg)\,dx = \int_{x_1}^{x_2} f'g\,dx + \int_{x_1}^{x_2} fg'\,dx$$

Putting these together

$$\int_{x_1}^{x_2}\frac{df}{dx}g\,dx = f(x)g(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} f\frac{dg}{dx}\,dx$$

Want to set $\frac{dI}{d\alpha} = 0$.

The pinnacle of the α cleverness is that this is to hold for <u>all</u> η. So, what can we say when

$$\int_{x_1}^{x_2} g(x)\eta(x)\,dx = 0$$

for all η? S'pose $g \neq 0$ and choose η positive whenever g is and similarly for negative values.

Then $g(x)\cdot\eta(x) > 0$ ← always! and

$$\int_{x_1}^{x_2} g(x)\eta(x)\,dx \neq 0,$$

a contradiction! $\Rightarrow \boxed{g(x) = 0}$.

<u>So</u>,

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \text{E-L eqns again!}}$$

That completes our 2nd proof.

Can go to examples (at last!).

So,

$$ds = \sqrt{R^2\left(\frac{d\phi}{dz}\right)^2 + 1}\, dz$$

and

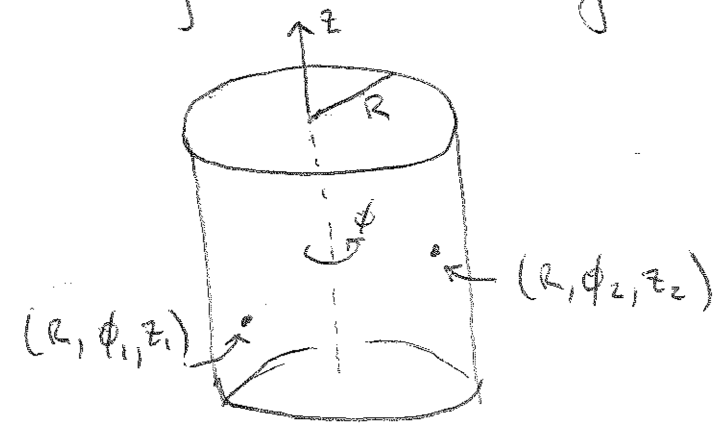
$$L = \int_{z_1}^{z_2} \underbrace{\sqrt{R^2(\phi')^2 + 1}}_{f(z,\phi,\phi') = f(\phi')}\, dz$$

E-L eqn:

$$\cancel{\frac{\partial f}{\partial \phi}}^{\,0} - \frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) = -\frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial \phi'} = \text{const.} \equiv C_1 \Rightarrow \frac{1}{2}\frac{1}{\sqrt{R^2\phi'^2+1}} \cdot 2R^2\phi' = C_1$$

Find the geodesics on a cylinder P5/5



Q: Shortest path connecting these pts?

Descripe as $\phi(z)$. Arclength

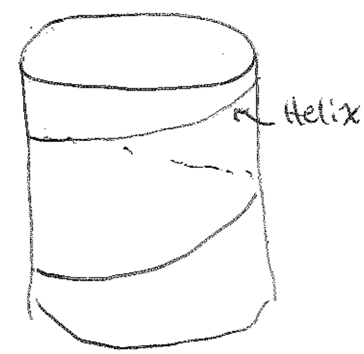
$$ds^2 = R^2 d\phi^2 + dz^2$$

or $R^4\phi'^2 = C_1^2(R^2\phi'^2 + 1)$

$$\Rightarrow (R^4 - C_1^2R^2)\,\phi'^2 = C_1^2$$

$$\Rightarrow \phi' = \text{const} \equiv C \quad \text{← call it}$$

Then $\phi(z) = Cz + k$ ← constant Helix eqn.



$$\left.\begin{aligned}\phi(z_1) &= Cz_1 + k\\ \phi(z_2) &= Cz_2 + k\end{aligned}\right\}$$

Solve C and k.