Day 7
Sept. 14th, 2011

Today's Outline

I. Lagrangian Formulation of Mechanics

II. Generalized Coordinates

III. Example of plane polar coordinates.

Introduce physical notation

	Indep. var.	Dep. var.	Function
Old (Math.)	x	$y(x)$	$f(x, y, y')$
New (Physics)	t (time)	$x(t), \vec{r}(t), q(t)$	$\mathcal{L}(q, \dot{q}, t)$

$\underbrace{x(t), \vec{r}(t), q(t)}_{\text{position as a function of time.}}$

x and $\vec{r} = (x, y, z)$ emphasize Cartesian coords

$\mathcal{L}(q, \dot{q}, t)$: New function called Lagrangian

The Lagrangian function is defined as

$$\mathcal{L} = T - U$$

where T is the kinetic energy

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and U is the potential energy

$$U = U(\vec{r}) = U(x, y, z).$$

Our Variational Principle

Hamilton's Principle

The actual path a particle follows btwn two pts 1 and 2 in a given time interval, t_1 to t_2, is s.t. the <u>action</u> integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

is stationary (i.e. extremized) when taken along this path.

We've already **emphasized** that this principle embodies a new, <u>independent</u>, axiom for mechanics. However, it'd be nice to see that it's equivalent to Newton's mechanics.

The Euler-Lagrange equations for the action (in 3D space w/ Cartesian coord.s $\vec{r} = (x, y, z)$) are

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \qquad \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = 0$$

and similarly for z.

Calculate x eq.;

$$\frac{\partial \mathcal{L}}{\partial x} = \cancel{\frac{\partial T}{\partial x}}^{\,0} - \frac{\partial U}{\partial x} = \underbrace{F_x}_{\text{force derivable from potential}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}} - \cancel{\frac{\partial U}{\partial \dot{x}}}^{\,0}\right)$$

$$= \frac{d}{dt}(m\dot{x}) = \frac{d}{dt}(p_x)$$

$$= m\ddot{x}$$

So the E.-L. equation implies

$$\text{E.-L.} \Rightarrow F_x = m\ddot{x}$$

Newton Eq! Can run argument backwards (line by line) to find Newton $\Rightarrow$ E.-L. eq.s.

Equivalent!

<u>Advantages of the Lagrangian Formulation:</u>

1. Lagrange's Equations, unlike Newton's, take the same form in all coordinate systems.
2. Lagrangian approach eliminates the forces of constraint.
3. Symmetries play an exquisite dual role in this formulation.

In the next few **days** we'll explore each of these advantages.

II We may want to use coord.s other than $\vec{r} = (x, y, z)$, e.g. (r, θ, ϕ), (ρ, ϕ, z) etc. We call coord.s (q_1, q_2, q_3) <u>generalized coordinates</u> if they are invertibly related to $\vec{r}$, i.e.

$$q_i(\vec{r}) \quad i = 1, 2, 3$$

and $\vec{r} = \vec{r}(q_1, q_2, q_3)$.

Using $\vec{r} = \vec{r}(q_1, q_2, q_3)$ we can write $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$ in terms of (q_1, q_2, q_3) and $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ i.e.

$$\dot{x} = \dot{x}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) \text{ etc.}$$

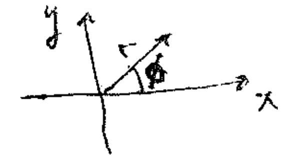
Then we can rewrite $\mathcal{L}$ in terms of the same quantities

$$\mathcal{L} = \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$$

$$= \mathcal{L}\left(\vec{r}(q_1, q_2, q_3), \dot{\vec{r}}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)\right).$$

<u>Example</u>: Plane polar coord.s (2D)

$$x = r\cos\phi \qquad y = r\sin\phi$$



$$r = \sqrt{x^2 + y^2}$$

$$\tan\phi = y/x \quad \leftarrow \text{use carefully!}$$

Invertible? Not everywhere! How do I check? Jacobian:

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \det\begin{pmatrix} c\phi & -rs\phi \\ s\phi & rc\phi \end{pmatrix}$$

$$= rc^2\phi + rs^2\phi = r$$

Non-singular as long as $r \neq 0$.

The action integral becomes

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)\, dt$$

and from this action we obtain the E-L equations

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) \quad \text{and sim. for } q_2, q_3.$$

These have exactly the same form as they did in Cartesian coord.s.

<u>Subtlety</u>: Above we found

$$\text{E.-L.} \Rightarrow \quad F_x = \frac{d}{dt}(P_x)$$

and said "Aha! Newton's law!"

But, this is only true in an inertial frame. So, the E.-L. equations derived from $\mathcal{L}$ only imply Newton's laws if you originally write down $\mathcal{L}$ in an <u>inertial frame</u>. Only then can you switch to any generalized coords you like.

<u>Example</u>:

III A particle moving in the plane described by polar coordinates, (r, ϕ).

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \frac{d}{dt}(r c\phi) = \dot{r} c\phi + r(-s\phi)\dot{\phi}$$

$$\dot{y} = \frac{d}{dt}(r s\phi) = \dot{r} s\phi + r c\phi \dot{\phi}$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 c^2\phi + r^2 s^2\phi \dot{\phi}^2 - 2\dot{r} r c\phi s\phi \dot{\phi}$$
$$+ \dot{r}^2 s^2\phi + r^2 c^2\phi \dot{\phi}^2 + 2 r \dot{r} c\phi s\phi \dot{\phi}$$
$$= \dot{r}^2(c^2\phi + s^2\phi) + r^2\dot{\phi}^2(c^2\phi + s^2\phi) = \dot{r}^2 + r^2\dot{\phi}^2.$$

This has an important practical significance not mentioned in your book. Often it is clear how to write your potential energy in the generalized coords. But almost always you want to start from

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and substitute $\vec{r}(\xi_1, \xi_2, \xi_3)$ to derive $T(\xi_i, \dot{\xi}_i)$ $i = 1, 2, 3$.

$U = U(r, \phi)$. So,

$$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi).$$

$$\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m \ddot{r}$$

$$\Rightarrow \quad -\frac{\partial U}{\partial r} = F_r = m(\ddot{r} - r\dot{\phi}^2).$$
$$= m a_r$$

So simple compared to the usual $\underbrace{\text{derivation of } a_r}$!
(see next time for this)