Day 8
Sept. 16th, 2015

Today's Outline:

I. Example of plane polar coordinates continued.

II Define generalized forces and momenta

III Constraints: an example.

I. Example: Plane Polar coord.s

Last time we started to find the Euler-Lagrange eq.s for a particle moving in the plane and described with plane polar coord.s.

Why? Plane polar coord.s are an example of generalized coord.s. If our system had "circular" symmetry they would be a good choice. We found

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2).$$

The potential in these coord.s is

$$U = U(r,\phi).$$

So,

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r,\phi)$$

and we can calculate:

$$\frac{\partial \mathcal{L}}{\partial r} = m r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \cancel{\dot{p}_r} = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

Now, $-\partial U/\partial r = F_r$ and so,

$$F_r = m\ddot{r} - mr\dot{\phi}^2 = m(\ddot{r} - r\dot{\phi}^2) = m a_r$$

So simple compared to the usual derivation of a_r!

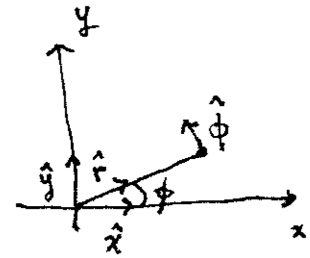
Let's review this derivation:

Write $\vec{r} = r\hat{r}$ so that

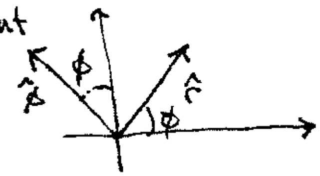
$$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

and

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$



Want to find $a_r = \hat{r}$-component of $\vec{a} = \ddot{\vec{r}}$. So, calculate



$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

and we need $\dot{\hat{r}}$. Well

$$\dot{\hat{r}} = -s\phi\,\dot{\phi}\,\hat{x} + c\phi\,\dot{\phi}\,\hat{y}$$
$$= \dot{\phi}(-s\phi\,\hat{x} + c\phi\,\hat{y}) = \dot{\phi}\,\hat{\phi}$$

So,

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

and

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}}.$$

We need $\dot{\hat{\phi}}$,

$$\dot{\hat{\phi}} = -c\phi\,\dot{\phi}\,\hat{x} - s\phi\,\dot{\phi}\,\hat{y}$$
$$= \dot{\phi}(-c\phi\,\hat{x} - s\phi\,\hat{y}) = \dot{\phi}(-\hat{r}).$$

Then

$$\vec{a} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + 2\dot{r}\dot{\phi}\hat{\phi}$$

$$\Rightarrow \quad a_r = \ddot{r} - r\dot{\phi}^2 \quad \checkmark$$

The Euler-Lagrange Equations are an <u>amazing computational gift</u>.

Let's do the ϕ component too:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial U}{\partial \phi} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(m r^2 \dot{\phi}\right)$$

So,

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}(m r^2 \dot{\phi}).$$

<u>Trick question</u>: What is $-\partial U/\partial \phi$?

It is not F_ϕ. Because $F_\phi = \hat{\phi}$-component of $-\vec{\nabla}U$ and in polar coord.s

$$\vec{\nabla}U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi}.$$

<u>So</u>,

$$F_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi}$$

This in turn implies

$$-\frac{\partial U}{\partial \phi} = r F_\phi = \text{torque} = \underset{\text{call it}}{\Gamma}$$

Meanwhile,

$$m r^2 \dot{\phi} = \underset{\text{moment of inertia}}{I}\ \underset{\text{angular velocity}}{\omega} = \underset{\text{angular momentum}}{L}$$

So, the ϕ E.-L. equation says that

$$\underset{\text{torque}}{\Gamma} = \frac{dL}{dt}$$

torque = time derivative of angular momentum!

The E.-L. equation automatically "knew" that ϕ was an angular coordinate.

II. This observation leads us to make two reasonable definitions. The E.-L. eq.s for generalized coord.s q_i $(i=1,\dots,n)$ are

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad (i=1,\dots,n)$$

We define

$$\frac{\partial \mathcal{L}}{\partial q_i} = (\text{ith component of a generalized force})$$

for example

$$= \underbrace{\text{a force, or a torque,}}_{\text{These are the most common cases}} \text{ or } \dots$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (\text{ith component of a generalized momentum})$$

for example

= a momentum or an angular momentum or ...

Last time we saw that the E.-L. equations took the same form in every coordinate system. Here we've also seen that they're very efficient. One can't help but ask: why?(!) This is a broad question but one answer is that $\mathcal{L}$ is a scalar. This is what allowed us to change coordinates and still end up with

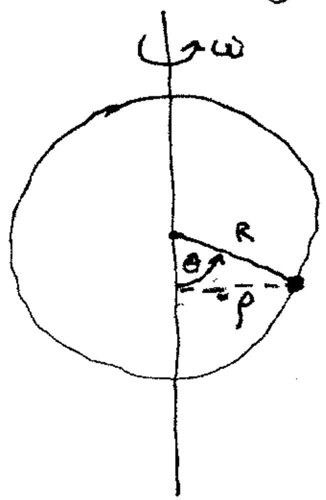
$S = \int \mathcal{L}\, dt$ and the E.-L. equations.

only write this half

In modern physics we exploit this fact to the hilt; we write down every known scalar consistent with the symmetries of the system and throw them into $\mathcal{L}$. This has become a central principle for constructing physical theories. More on symmetry next week.

III Constraints: an example

Bead on a Spinning Wire Hoop



The bead is constrained to move along the wire. In particular if you know $\theta(t)$ then you can find $x(t)$, $y(t)$ and $z(t)$ (given, say, that the hoop is in the xz-plane at $t=0$). Try it. We say that the bead has one <u>degree of freedom</u>.

Let's find the Lagrangian. We choose an inertial frame, say the room our apparatus is in (Note: a frame rotating w/ the hoop would not work!) Then the bead has velocity $R\dot{\theta}$ tangential to the hoop and $\rho\dot{\phi} = \rho\omega$ normal to it (into the page in our picture). So,

$$T = \frac{1}{2}m\left((R\dot{\theta})^2 + (\rho\omega)^2\right)$$

'cause $\rho = R\sin\theta$

$$= \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\omega^2\right)$$

Gravitational potential energy is

$$U = mg\left(R - R\cos\theta\right)$$

$$= mgR\left(1-\cos\theta\right)$$

So

$$\mathcal{L} = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\,\omega^2\right) - mgR\left(1-\cos\theta\right)$$

and E.-L. eq. is

$$\frac{\partial\mathcal{L}}{\partial\theta} = mR^2\sin\theta\cdot\cos\theta\,\omega^2 - mgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2\ddot{\theta}$$

$$\Rightarrow \; mR^2\left(s\theta c\theta\, \omega^2 - \frac{g}{R} s\theta\right) = mR^2\ddot{\theta}$$

$$\Rightarrow \; \boxed{\ddot{\theta} = \left(c\theta\, \omega^2 - g/R\right) \cdot s\theta}$$

But wait, was I justified in applying the E.-L. eq.?

We've only shown that it works for unconstrained systems. You should check this equation with Newton's equations.

N particles in 3D 3N D.O.F.

When the # of D.O.F. of N particles in 3D is less than 3N, we say system is constrained.

Next time: We'll show whether the E.-L. equations apply to constrained systems.

Setup: Definition of degrees of freedom (D.O.F.) in general:

\# of D.O.F. = # of coords that can be independently varied in a small displacement.

e.g. pendulum 1 D.O.F.
double pendulum 2 D.O.F.