Day 7

Sept. 18th, 2015

Outline

I Finish example

II Constraints

The E-L equation is

$$\frac{\partial \mathcal{L}}{\partial \theta} = mR^2 s\theta c\theta \omega^2 - mgR s\theta$$

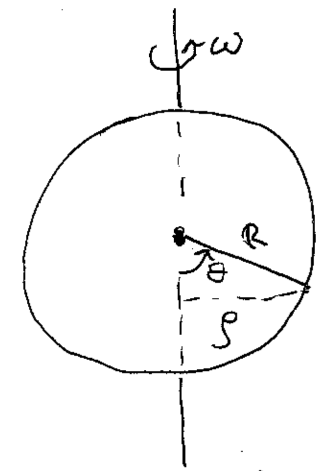
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mR^2 \dot{\theta}\right) = mR^2 \ddot{\theta}$$

$$\Rightarrow mR^2\left(s\theta c\theta \omega^2 - \frac{g}{R} s\theta\right) = mR^2 \ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} = \left(c\theta \omega^2 - g/R\right) s\theta}$$

But wait, was I justified in applying the E-L eqn.?

I Recall



Last time we derived the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} mR^2(\dot{\theta}^2 + s^2\theta\omega^2) - mgR(1 - c\theta)$$

We've only shown that it works for unconstrained systems (check our $\ddot{\theta}$ eqn using Newton's laws.)

II The 2nd great advantage of the Lagrangian formulation is its seemless incorporation of constraints. Let's show this.

<u>Setup</u>: Def. of degrees of freedom (...) ... general:

# of D.O.F. = # of coord.s that can be independently varied in a small displacement.

E.g. pendulum 1 D.O.F.

N particles in 3D 3N D.O.F.

When the # of D.O.F. of N particles in 3D is less than 3N, we say the system is constrained,

Often

D.O.F. = # generalized coord.s used to describe system

in this case we say the constraints are holonomic

(Beware other defn.s exist.)

We will focus on holonomic constraints

Goal: prove that E.-L. eqn.s hold even for systems with constraints:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad i=1,\dots,n$$

n = # gen. coord.s = # D.O.F.

Proof has two steps:

Step 1: The "what's so special about (holonomic) constraint forces?" step. Answer: They do no work.

Step 2: The "Oh, duh, it's exactly ... as before" step.

More intuitively holonomic constraints are geometric conditions we know the system will obey but for which we don't know the force equations.

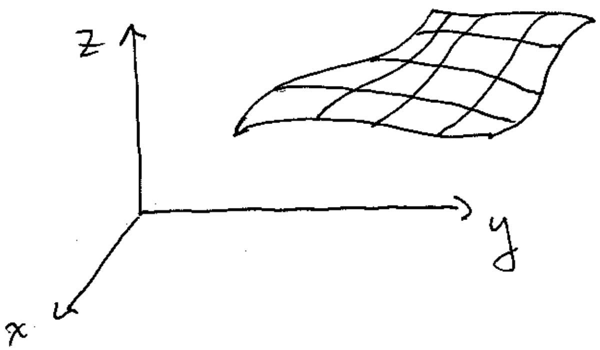
E.g. the bead on a hoop example.

Or a particle confined to move on a curved surface. We'll take this as our motivating example (see below)

...

there are two kinds of [illegible]

applied forces (e.g. gravity) and Constraint forces (e.g. the normal force due to the surface). Your book calls the applied forces "non-constraint" forces.



← coord.s on surface q_1 and q_2.

D.O.F. = 2

Give [illegible]

$\vec{F}$ = applied forces

$\vec{F}_{cstr}$ = constraint forces

$$\vec{F}_{tot} = \vec{F} + \vec{F}_{cstr}$$

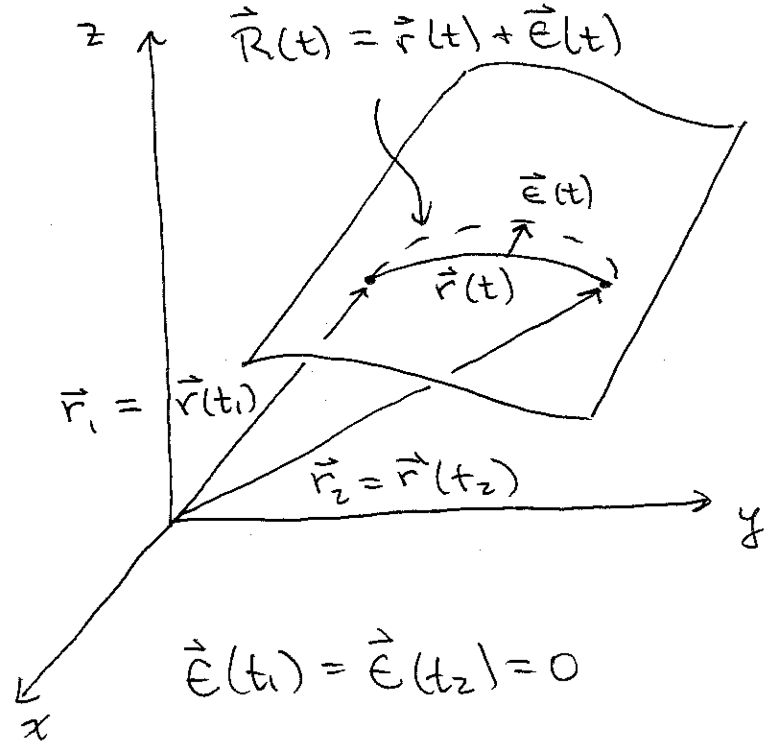
We will assume all the applied forces are derivable from a potential,

$$\vec{F} = -\vec{\nabla} U(\vec{r}, t)$$

Let

$$\mathcal{L} = T - U$$

and note that this excludes the constraint forces because of our definition of U. What happens to Hamilton's principle when we only allow variations over paths contained in the surface?



$$\vec{\epsilon}(t_1) = \vec{\epsilon}(t_2) = 0$$

Let, $S = \int_{t_1}^{t_2} \mathcal{L}(\vec{R}, \dot{\vec{R}}, t)\, dt$,

$S_0 = \int_{t_1}^{t_2} \mathcal{L}(\vec{r}, \dot{\vec{r}}, t)\, dt$,

and change in integral S due to change in path

$\delta S = S - S_0$

This boils down to

$$\delta \mathcal{L} = \mathcal{L}(\vec{R}, \dot{\vec{R}}, t) - \mathcal{L}(\vec{r}, \dot{\vec{r}}, t)$$

$$= \mathcal{L}(\vec{r} + \vec{\epsilon},\ \dot{\vec{r}} + \dot{\vec{\epsilon}}, t) - \mathcal{L}(\vec{r}, \dot{\vec{r}}, t)$$

and $O(\epsilon^2)$ means "terms of order ϵ^2 or higher. Then

$$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L}\, dt$$

$$= \int_{t_1}^{t_2} \left[m \dot{\vec{r}} \cdot \dot{\vec{\epsilon}} - \vec{\epsilon} \cdot \vec{\nabla} U \right] dt$$

integrate by parts

$$= - \int_{t_1}^{t_2} \vec{\epsilon} \cdot \left[m \ddot{\vec{r}} + \vec{\nabla} U \right] dt$$

As well,

$$m \ddot{\vec{r}} = \vec{F} = \vec{F} + \vec{F}$$

Now,

$$\mathcal{L} = \tfrac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r}, t)$$

so,

$$\delta \mathcal{L} = \tfrac{1}{2} m \left[(\dot{\vec{r}} + \dot{\vec{\epsilon}})^2 - \dot{\vec{r}}^2 \right] - \left[U(\vec{r} + \vec{\epsilon}, t) - U(\vec{r}, t) \right]$$

$$= m \dot{\vec{r}} \cdot \dot{\vec{\epsilon}} - \vec{\epsilon} \cdot \vec{\nabla} U + O(\epsilon^2)$$

The 2^{nd} term comes from multi-dimensional Taylor thm:

$$U(\vec{r} + \vec{\epsilon}, t) = U(\vec{r}, t) + \vec{\epsilon} \cdot \vec{\nabla} U(\vec{r}, t) + \cdots$$

$$= -\vec{\nabla} U + \vec{F}_{cstr}$$

$$\Rightarrow \delta S = - \int_{t_1}^{t_2} (\vec{\epsilon} \cdot \vec{F}_{cstr})\, dt,$$

but $\vec{\epsilon} \cdot \vec{F}_{cstr} = 0$ ($\vec{\epsilon}$ is tangent to constraint surface

$$\Rightarrow \boxed{\delta S = 0}$$

Hamilton's principle still holds!

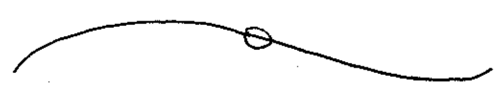
Now, in our case of a 2D constraint surface in 3D space, the coord.s on the surface, q_1 & q_2, are independent. This means that we can write

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2, t)\, dt$$

because all other coord.s depend on q_1 and q_2. Our argument

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad (i=1,2).$$

The constraints have been completely absorbed into our choice of generalized coord.s.



We've done it. Lagrangian Mechanics provides a completely independent formulation of Mechanics that is equivalent to Newton's.

From here forward we can use whatever formulation is most convenient. I will do this.

That

$$\delta S = 0$$

is another way of saying that S is stationary ~~for~~ the physical path (similar to $df = 0$ in calculus). But, we know that the conditions for

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2, t)\, dt$$

to be stationary are: