Homework 13

Due Friday, December 14th at 5pm

You will need to consult your class notes for this problem set.

1. Find the curvature of the ellipse described by $\vec{r}(t) = (2\cos(t), \sin(t))$.

   (a) Here it is not a good strategy to first find the arclength of the ellipse. This is because the arclength is a complicated function of any parameter (it is the central example of an elliptic integral and motivates the name for this class of integrals). Instead, first find any tangent vector to the curve and then normalize that tangent vector to get the unit tangent vector.

   (b) Starting from our definition of the curvature vector $\vec{\kappa}$, use the chain rule to express this in terms of the time derivative of the unit tangent vector and the time derivative of the parametrized curve $\dot{\vec{r}}(t)$.

   (c) Use your result from (b) to compute the curvature of this ellipse.

   (d) Sketch the ellipse. Where along the curve is the curvature largest according to your result from (c)? Sketch these points on your curve. Does the result make sense?

2. A two-dimensional isotropic harmonic oscillator has the traditional Lagrangian

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2).$$

   where as usual $\dot{x} \equiv dx/dt$ and similarly for $\dot{y}$.

   (a) What is the extended Lagrangian L for this system? Is this Lagrangian homogeneous of degree one in the parametrized velocities?

   (b) Find the generalized momenta for each of the coordinates of the extended Lagrangian L.

   (c) Use the extended Lagrangian to find the extended equations of motion? Is energy conservation again one of these equations? If so, what symmetry indicates that this should be so?

   (d) Assuming that you started with a parameter σ, change your parametrization to $\tilde{\sigma} = f(\sigma)$, where f is an arbitrary smooth function. Find the new extended Lagrangian after this change of parametrization. Choosing whichever equation of motion you like, check that this new extended Lagrangian gives the same equaiton of motion as before the change of parametrization. (This is the statement I claimed in class, but didn't prove.)

3. Show that the points $(\frac{1}{2}, 1, 1)$, $(1, \frac{1}{3}, \frac{4}{3})$, and $(2, -1, 2)$ of the real projective plane are collinear, and find an equation for the line containing them.