Mechanics
Day 2

<u>Today</u>

I Last time

II Standard guess

III Damped Oscillations

IV Regimes

I. Dot notation:

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \ddot{x} = \frac{d^2x}{dt^2}$$

Oscillations are ubiquitous because the PE is generically quadratic in the vicinity of a stable equilibrium

Second derivative test:

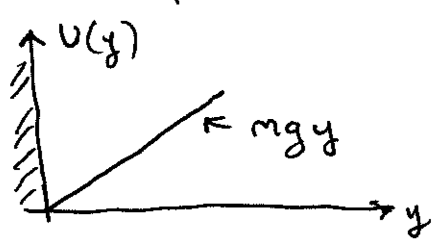
stable: $\frac{d^2U}{dx^2} > 0$ unstable: $\frac{d^2U}{dx^2} < 0$

This means we can find the effective spring constant using

$$k = U''(x_{eq}).$$

There are exceptions, e.g. the bouncy ball

below the floor PE is ∞



Here we can't Taylor expand because $U(y)$ is not differentiable at $y=0$.

II. We will change perspective in this course

$$F = ma = -kx$$

$$\downarrow \quad m\frac{d^2x}{dt^2} = -kx$$

$$\downarrow \quad m\ddot{x} = -kx$$ viewed as a differential eqn.

We will learn techniques for writing

down eqn.s of motion (EOM), <u>but</u> <u>also</u> for solving them.

<u>Terminology</u>:

Ordinary diff. eqn. (ODE): only ordinary derivatives

Partial diff. eqn. (PDE): partial derivatives

Order of diff. eqn.: the highest derivative that appears.

Take the simple harmonic oscillator as an example, it's a 2^{nd} order ODE. The most important method of solution is the "standard guess":

$$x(t) = e^{rt}$$

The EOM is

$$m\ddot{x} = -kx.$$

With this guess we have

$$\dot{x} = re^{rt}, \quad \ddot{x} = r^2 e^{rt}$$

so, $m r^2 \cancel{e^{rt}} = -k \cancel{e^{rt}}$

and we've reduced the ODE to an algebraic eqn:

$$\Rightarrow r = \pm i\sqrt{\frac{k}{m}} \equiv \pm i\omega_0$$

General solution is a linear combo (superposition):

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

constants

A 2^{nd} order ODE always has a general solution depending on 2 consts.

Physically, it corresponds to the fact that you need boundary conditions:

$x(0) \equiv x_0 = C_1 + C_2$ and

$\dot{x}(0) \equiv v_0 = i\omega_0 C_1 - i\omega_0 C_2.$

<u>Euler's exquisite creation</u>:

$$e^{i\pi} + 1 = 0$$

Generally,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Amazing secret: $e^{i\theta} = c\theta + is\theta.$

Can use this to derive all of trigonometry!

<u>Example:</u> $e^{i2\theta} = C2\theta + iS2\theta$

$$= (e^{i\theta})^2 = (C\theta + iS\theta)^2$$

$$= C^2\theta - S^2\theta + 2i\,C\theta \cdot S\theta$$

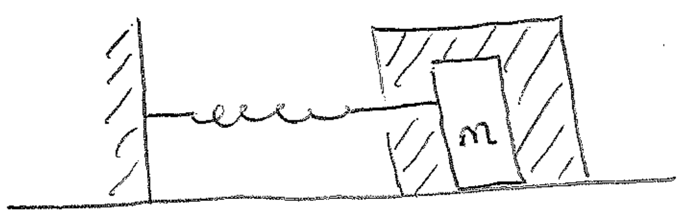
Real parts equal implies

$$C2\theta = C^2\theta - S^2\theta = 2C^2\theta - 1 \quad \text{etc.}$$

III Damped Harmonic Oscillations

Hooke's law: $F_{spring} = -kx$

Immerse the mass in a tank of viscous medium.



<u>Challenge:</u> For the duration of this course never look up a trig. formula.

<u>Challenge 2:</u> Figure out four distinct ways to write the general solution to

$$m\ddot{x} = -kx$$

using complex numbers (solution in your text)

Assume force is proportional to velocity and opposite in direction

$$f_{viscous} = -bv$$

(e.g. neglecting turbulence)

Newton's 2nd law:

$$F_{net} = F_{spring} + f_{viscous} = ma = m\ddot{x}.$$

So, $m\ddot{x} = -kx - bv$

or

$$m\ddot{x} + b\dot{x} + kx = 0$$

or $\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0$

Recall, $\omega_0 = \sqrt{k/m}$ and introduce shorthand $\beta = \frac{b}{2m}$ so that

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Solve for $x(t)$? Use standard guess, $x = e^{rt}$

$$\dot{x} = re^{rt}, \quad \ddot{x} = r^2 e^{rt}$$

$$r^2 \cancel{e^{rt}} + 2\beta r \cancel{e^{rt}} + \omega_0^2 \cancel{e^{rt}} = 0$$

$$\Rightarrow r^2 + 2\beta r + \omega_0^2 = 0$$

$$\Rightarrow r = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$$

$$= -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Call these

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}, \quad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Important differences in this solution depending on regime of the parameters β, ω_0.

IV <u>Three Regimes</u>

(1) Overdamped: $\beta > \omega_0$.

Then $B \equiv \sqrt{\beta^2 - \omega_0^2}$ is real and the general solution is

$$x(t) = C_1 e^{(-\beta + B)t} + C_2 e^{(-\beta - B)t}$$

Both terms are damping exponentials (Note $\beta > B > 0$).

[Aside: For exp.s the "characteristic time", τ, is the time it takes to get to $1/e$ of its original value.]

In the overdamped regime there are two characteristic times

$$\tau_1 = \frac{1}{\beta - B} \; ; \quad \tau_2 = \frac{1}{\beta + B}.$$

In fact, for Huge damping $\beta \approx B$ and $\tau_1 \to \infty$.

(2) Underdamped: $\beta < \omega_0$.

Then $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ is real

$$r_{1,2} = -\beta \pm i\omega_1$$

and the general Soln. is

$$x(t) = C_1 e^{(-\beta + i\omega_1)t} + C_2 e^{(-\beta - i\omega_1)t}$$

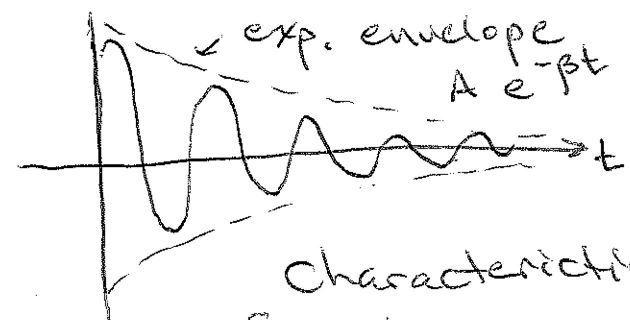
$$= e^{-\beta t}\left(C_1[\cos\omega_1 t + i\sin\omega_1 t] + C_2[\cos\omega_1 t - i\sin\omega_1 t]\right)$$

$$= e^{-\beta t}\left(D_1 \cos\omega_1 t + D_2 \sin\omega_1 t\right)$$

with $D_1 = C_1 + C_2$, $D_2 = i(C_1 - C_2)$

$$\boxed{x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)}$$

Sinusoidal oscillations w/ exp. decreasing amplitude $Ae^{-\beta t}$



Characteristic time for damping $\tau = 1/\beta$

Two times at work here: damping $\tau = 1/\beta$; and the Period of the oscillations

$$T = \frac{2\pi}{\omega_1}$$